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On wave propagation in repetitive structures: Two forms of transfer matrix

A.K. Ashari [b], N.G. Stephen [a, b, *]

[a] Mechanical Engineering, Faculty of Engineering and Physical Sciences, University of Southampton, Highfield, Southampton, SO17 1BJ, UK

[b] University of Southampton Malaysia, 3, Persiaran Canselor 1, Kota Ilmu Educity@Iskandar, 79200 Iskandar Puteri, Johor, Malaysia

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Two forms of dynamic transfer matrix are derived for a one-dimensional (beam-like) repetitive pin-jointed structure with point masses located at nodal cross-sections, the displacement-force transfer matrix **G**, and the displacement-displacement transfer matrix, **H**. Similarity matrices are introduced to relate **G** and **H**, together with their respective metrics. Symplectic orthogonality relationships for the eigenvectors of both **G** and **H** are derived, together with relationships between their respective sets of eigenvectors. New expressions for the group velocity are derived. For repetitive structures of finite length, natural frequency equations are derived employing both **G** and **H**, including phase-closure and the direct application of boundary (end) conditions. Besides an exposition of the theory, some familiar but much new, the focus of the present paper is on the relationships between the two forms of transfer matrix, including their respective (dis)advantages. Numerical results, together with further theory necessary for interpretation, are presented in companion papers.

1. Introduction

Repetitive or periodic structures consist of a cell which spatially repeats in one-, two- or three-dimensions. Each cell is connected to another in a regular pattern to form the complete structure. Such constructions are widely employed in engineering, and include rail track, turbine blade assemblies (bladed discs), building frameworks, cranes, aircraft fuselages, trusses and honeycomb panels. Since the manufacture and construction of such structures can also be a repetitive process, they represent a cost effective design solution for many engineering applications. Early contributions are described in Refs. [1–7]. The joints between the structural members can be designed so that they allow additional degrees of freedom, providing the possibility of a change in structural shape [8], or to become a deployable mechanism/structure [9]. Furthermore, repetitive structures portray symmetrical features and often have an aesthetically pleasing appearance.

The present work is concerned with one-dimensional (beam-like) repetitive structures. When periodicity is taken into account, the static and dynamic analysis of an entire structure can be reduced to the analysis of a single repeating cell, together with boundary (end) conditions if the structure is not of infinite extent; equivalent continuum properties can be

* Corresponding author. Mechanical Engineering, Faculty of Engineering and Physical Sciences, University of Southampton, Highfield, Southampton, SO17 1BJ, UK.

E-mail addresses: aka104@soton.ac.uk (A.K. Ashari), ngs@soton.ac.uk (N.G. Stephen).

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Nomenclature

A	cross-sectional area
c_g	group velocity
$\mathbf{d}$, $\mathbf{D}$, D	displacement vectors, components
E	Young's modulus of elasticity
$\mathbf{f}$, $\mathbf{F}$, F	force vectors, components
$\mathbf{G}$	displacement-force transfer matrix
$\mathbf{H}$	displacement-displacement transfer matrix
i, i, $\mathbf{I}$	$\sqrt{-1}$, index, identity matrix
j, $\mathbf{J}$	index, metric matrix
k, $\mathbf{K}$	bar stiffness, stiffness matrix
L, L, $\mathbf{L}$	left, bar length, similarity matrix
m, $\mathbf{m}$	mass, mass matrix
n, N, $\mathbf{N}$	index, number of cells, similarity matrix
R, $\mathbf{R}$	right, reflection matrix
$\mathbf{s}$	generic state vector
t, $\mathbf{T}$	time, generic transfer matrix
$\mathbf{V}$	(right) eigenvector of $\mathbf{G}$, matrix of eigenvectors
$\mathbf{w}$, $\mathbf{W}^{\mathrm{T}}$	wave amplitude vector, left eigenvector of $\mathbf{G}$
$\mathbf{X}$	(right) eigenvector of $\mathbf{H}$, matrix of eigenvectors
$\mathbf{Y}^{\mathrm{T}}$	left eigenvector of $\mathbf{H}$
δ	decay rate constant
φ	phase change constant, or wavenumber
λ, $\mathbf{\Lambda}$	eigenvalue, matrix of eigenvalues
ω, $\mathbf{\Omega}$	radian frequency, skew-symmetric matrix

determined for segmented structures such as trusses [10]. The primary approach is through the use of a transfer matrix $\mathbf{T}$, which relates state vector components on the right-hand side to those on the left-hand side of the cell, i.e. $\mathbf{s}_{\mathrm{R}} = \mathbf{T}\mathbf{s}_{\mathrm{L}}$. (Alternative analytical approaches, including the receptance method, are described in Mead's 1996 review paper [7].) An eigenvector of the transfer matrix describes a pattern of state vector components which is unique to within a scalar multiplier, λ. Translational symmetry demands that this pattern is preserved as one moves from the left-hand to the right-hand side of the cell, allowing one to write $\mathbf{s}_{\mathrm{R}} = \lambda\mathbf{s}_{\mathrm{L}}$; this immediately leads to the standard eigenvalue problem $\mathbf{T}\mathbf{s}_{\mathrm{L}} = \lambda\mathbf{s}_{\mathrm{L}}$, or $(\mathbf{T}-\lambda\mathbf{I})\mathbf{s}_{\mathrm{L}} = \mathbf{0}$. There are two forms of the transfer matrix $\mathbf{T}$ in frequent use: the first and more common [3,10–18,23–25] relates a state vector $\mathbf{s}$ of displacement and force components on either side of the cell, here presented as transfer matrix $\mathbf{G}$; the second and less common form [8,19–22] relates state vectors $\mathbf{s}$ of displacements at three consecutive nodal cross-sections of the complete structure, here presented as $\mathbf{H}$.

Both $\mathbf{G}$ and $\mathbf{H}$ can be determined from the stiffness matrix $\mathbf{K}$; the latter is symmetric for linear elastic displacements and, in turn, both transfer matrices are symplectic, that is they satisfy a relationship of the form $\mathbf{T}^{\mathrm{T}}\mathbf{\Omega}\,\mathbf{T} = \mathbf{\Omega}$, where $\mathbf{T}^{\mathrm{T}}$denotes the transpose of $\mathbf{T}$, and $\mathbf{\Omega}$ is a skew-symmetric matrix, known as the *metric*. For transfer matrix $\mathbf{G}$, the metric takes its simplest, canonical, form, and is written as $\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$. The significance of the metric is rooted in Hamiltonian mechanics. The metric in a Euclidean vector space is the *length* of a vector, calculated as the square root of the dot product of the vector with itself, an inner product. However, in a symplectic vector space, where the state vector consists of both displacement and force components, such an inner product has no physical meaning. Instead, a symplectic inner product, employing $\mathbf{J}$ as the metric, multiplies displacement with force which is work or energy; rather than *length*, it is an *area*, which is preserved during (here spatial) evolution. Ultimately, it implies conservation of energy.

For the static problem, the *force-displacement* transfer matrix $\mathbf{G}$ is perhaps the more appropriate, as one can readily identify force resultants; thus the decay modes associated with self-equilibrated loading (as anticipated by Saint-Venant's principle), the rigid body modes associated with zero force components, and the transmission modes associated with the force resultants of tension, bending moment and shearing force can be easily recognised. This static problem is characterised by multiple unity eigenvalues for the rigid body and transmission modes. In turn the transfer matrix cannot be diagonalised, but can only be reduced to a Jordan canonical block form; e.g. the principal vector describing tension is coupled with the eigenvector for a rigid-body displacement in the axial x-direction within a (2×2) block [10].

For wave propagation, the *displacement-displacement* transfer matrix $\mathbf{H}$ is perhaps the more appropriate, as waves are naturally described in terms of their displacement characteristics, e.g. extensional, flexural, thickness-shear, rather than force resultants.