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Computing the average root number of an elliptic surface

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By considering a one-parameter family of elliptic curves defined over $\mathbb{Q}$, we might ask ourselves if there is any bias in the distribution (or parity) of the root numbers at each specialization. From the work of Helfgott [8], we know (at least conjecturally) that the average root number of an elliptic curve defined over $\mathbb{Q}(T)$ is zero as soon as there is a place of multiplicative reduction over $\mathbb{Q}(T)$ other than $-\deg$.

In this paper, we are concerned with elliptic curves defined over $\mathbb{Q}(T)$ with no place of multiplicative reduction over $\mathbb{Q}(T)$, except possibly at $-\deg$. In [1], the authors classify all such one-parameter families of elliptic curves whose coefficients, in the parameter T, have degree less than or equal to 2; they also use the work of Helfgott to compute the average root number of two particular subfamilies. We complement the work in [1] by computing the average root number of one of these "potentially parity-biased" families and show that it is "parity-biased" infinitely-often.

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1. Introduction

Let E be an elliptic curve defined over $\mathbb{Q}$. For every prime p, let $\tilde{E}_p$ denote the reduction of E modulo p and set $a_p := p + 1 - \#\tilde{E}_p(\mathbb{F}_p)$, where $\#\tilde{E}_p(\mathbb{F}_p)$ denotes the number of $\mathbb{F}_p$-points on $\tilde{E}_p$. The L-series associated to E is defined by the Euler product

$$L(s,E) := \prod_{\substack{p \text{ prime} \\ p \mid \Delta}} (1 - a_p p^{-s})^{-1} \prod_{\substack{p \text{ prime} \\ p \nmid \Delta}} (1 - a_p p^{-s} + p^{1-2s})^{-1},$$

where Δ is the discriminant of E. It is well known that the product defining $L(s,E)$ converges and gives rise to an analytic function, provided $\Re(s) > \frac{3}{2}$. The Modularity Theorem [17] tells us that much more is true; namely,

$$\Lambda(s,E) := N_E^{\frac{s}{2}} (2\pi)^{-s} \Gamma(s) L(s,E)$$

has an analytic continuation to the entire complex plane and satisfies the functional equation

$$\Lambda(s,E) = w\Lambda(2-s,E),$$

for some $w = w_E = \pm 1$, where $N_E = N_{E/\mathbb{Q}}$ is the conductor of E and where $\Gamma(s) := \int_0^\infty t^{s-1} \mathrm{e}^{-t}\, dt$ is the Gamma function. We call w the *root number of* E.

In this paper, we use the techniques developed by Rizzo [11] and generalized by Helfgott [8] to compute the average root number of an explicit family of elliptic curves defined over $\mathbb{Q}$. By a family of elliptic curves defined over $\mathbb{Q}$, we mean an elliptic curve defined over $\mathbb{Q}(T)$; equivalently, it is a one-parameter family of elliptic curves given by a Weierstrass equation of the form

$$\mathcal{F} : y^2 = x^3 + a_2(T)x^2 + a_4(T)x + a_6(T),$$

for some $a_2(T), a_4(T), a_6(T) \in \mathbb{Z}[T]$. For every $t \in \mathbb{Z}$, we let $\mathcal{F}(t)$ denote the specialization of $\mathcal{F}$ at t and note that $\mathcal{F}(t)$ defines an elliptic curve for all but finitely-many t. Moreover, the map which sends $\mathcal{F}$ to $\mathcal{F}(t)$ is injective for all but finitely-many t (Silverman's Specialization Theorem, [14]). From here, we let

$$\varepsilon_{\mathcal{F}}(t) := \begin{cases} \text{the root number of } \mathcal{F}(t) & \text{if } \mathcal{F}(t) \text{ is an elliptic curve,} \\ 0 & \text{otherwise,} \end{cases}$$

and define the *average root number of* $\mathcal{F}$ *over* $\mathbb{Z}$ by

$$\mathrm{Av}_{\mathbb{Z}}(\varepsilon_{\mathcal{F}}) := \lim_{T\to\infty} \frac{1}{2T} \sum_{|t| \le T} \varepsilon_{\mathcal{F}}(t),$$

provided the limit exists.