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PLANE MODEL-FIELDS OF DEFINITION, FIELDS OF DEFINITION, AND THE FIELD OF MODULI FOR SMOOTH PLANE CURVES

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Abstract. Let $C/\overline{k}$ be a smooth plane curve defined over $\overline{k}$, a fixed algebraic closure of a perfect field k. We call a subfield $k' \subseteq \overline{k}$ a *plane model-field of definition for* C if C descends to k' as a smooth plane curve over k', that is if there exists a smooth curve C'/k' defined over k' which is k'-isomorphic to a non-singular plane model $F(X,Y,Z)=0$ with coefficients in k', and such that $C' \otimes_{k'} \overline{k}$ and C are isomorphic. In this paper, we provide (explicit) families of smooth plane curves for which the three fields types; the field of moduli, fields of definition, and plane-models fields of definition are pairwise different.

1. Introduction

Let C/k be a smooth projective curve defined over a perfect field k. We say that C *descends to a subfield* $k' \subset k \subset \overline{k}$, where $\overline{k}$ is a fixed algebraic closure of a field k, if there exists a smooth projective curve C'/k' defined over k' such that $C' \otimes_{k'} k \cong C$. In this case, k' is called a *field of definition for* C. The intersection of all fields of definition for $C \otimes_k \overline{k}$ is called the *absolute field of moduli for* C and is denoted by k_C. Alternatively, the *relative field of moduli to the field extension* k/k' is commonly used in the literature, and it is defined to be the subfield $M_{k/k'}(C)$ fixed by the subgroup

$$\{\sigma \in \mathrm{Gal}(k/k') : C \cong_k {}^{\sigma}C\}.$$

The main relation between the absolute and the relative field of moduli is due to [8, Theorem 1.6.9]; k_C is a field of definition for C if and only if given any algebraically closed field $K \supseteq k$, and any subfield $k' \subseteq K$ with K/k' Galois, $M_{K/k'}(C \otimes_k K)$ is a field of definition for $C \otimes_k K$.

Finding fields of definition and/or fields of moduli of varieties is a long standing problem. It is also very common to ask when the field of moduli for a smooth projective curve is a field of definition. A necessary and sufficient condition (Weil's cocycle criterion of descent) for the field of moduli to be a field of definition was provided by Weil [17]. If the full automorphism group $\mathrm{Aut}(C \otimes_k \overline{k})$ is trivial, then this condition becomes trivially true and so the field of moduli needs to be a field of definition. It is also well known that if C has geometric genus $g=0$ then it is $\overline{k}$-isomorphic to the projective line $\mathbb{P}^1$, which is defined over k_0, the prime field of k (cf. [5, §1]). Moreover, if $g=1$, then the field of moduli is $k_0(j)$, where j is the modular invariant of C, and it is known that for characteristic $p \neq 2,3$, C is $\overline{k}$-isomorphic to a model defined over $k_0(j)$ (cf. [16, Chp. III, Proposition 1.4]). However, if $g>1$ and $\mathrm{Aut}(C \otimes_k \overline{k})$ is non-trivial, then Weil's conditions are difficult to be checked and so there is no guarantee that the field of moduli is a field of definition for $C \otimes_k \overline{k}$. Explicit examples of hyperelliptic curves over $\mathbb{C}$ not definable over their field of moduli were first provided by Earle [6], Shimura [15]. Later on, Huggins [8, 9] proved that a hyperelliptic curve of genus $g \geq 2$ defined over

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