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Stochastic Processes and their Applications 128 (2018) 3880–3896

stochastic processes and their applications

www.elsevier.com/locate/spa

Denseness of volatile and nonvolatile sequences of functions

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Received 11 July 2017; received in revised form 13 December 2017; accepted 4 January 2018
Available online 31 January 2018

Abstract

In a recent paper by Jonasson and Steif, definitions to describe the volatility of sequences of Boolean functions, $f_n : \{-1, 1\}^n \to \{-1, 1\}$ were introduced. We continue their study of how these definitions relate to noise stability and noise sensitivity. Our main results are that the set of volatile sequences of Boolean functions is a natural way "dense" in the set of all sequences of Boolean functions, and that the set of non-volatile Boolean sequences is not "dense" in the set of noise stable sequences of Boolean functions.

MSC: 60J27; 60K99; 05C81

Keywords: Volatility; Noise sensitivity; Noise stability; Boolean functions

1. Introduction

This paper will be concerned with the volatility of sequences of Boolean functions as defined in [3], and in particular with the relation between the set of volatile sequences of Boolean functions and the sets of noise stable and noise sensitive sequences of Boolean functions respectively (see e.g. [2]). All of these definitions can be said to describe aspects of the behaviour of the value of a Boolean function when its input evolves according to a simple Markov chain (X_t).

The Markov chain $(X_t^{(n)})_{t \geq 0}$, $X_t^{(n)} = (X_t^{(n)}(1), \ldots, X_t^{(n)}(n))$, with which we will be concerned will have $\{-1, 1\}^n$ as its state space. We define the Markov chain by letting each coordinate

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https://doi.org/10.1016/j.spa.2018.01.001
0304-4149/

update independently according to an exponential clock with rate one, setting the value at an updating coordinate to 1 with probability p_n and to -1 with probability $1 - p_n$. Clearly, the stationary measure π_{p_n} for this process will be $\{1 - p_n, p_n\}^n$, and whenever nothing else is written explicitly we will pick $X_0^{(n)}$ according to this measure. To stress the dependence of (p_n) we sometimes add p_n as a subscript to P, and write P_{p_n}, Cov_{p_n} etc.

If we compare the value of the process $(X_t^{(n)})$ at a coordinate i for $t = 0$ and $t = \varepsilon$, we get

$$\begin{cases} P_{p_n}\left[X_\varepsilon^{(n)}(i) = 1 \mid X_0^{(n)}(i) = 1\right] = e^{-\varepsilon} + (1 - e^{-\varepsilon})p_n \\ P_{p_n}\left[X_\varepsilon^{(n)}(i) = -1 \mid X_0^{(n)}(i) = 1\right] = (1 - e^{-\varepsilon})(1 - p_n) \\ P_{p_n}\left[X_\varepsilon^{(n)}(i) = -1 \mid X_0^{(n)}(i) = -1\right] = e^{-\varepsilon} + (1 - e^{-\varepsilon})(1 - p_n) \\ P_{p_n}\left[X_\varepsilon^{(n)}(i) = 1 \mid X_0^{(n)}(i) = -1\right] = (1 - e^{-\varepsilon})p_n. \end{cases}$$

Consequently, $X_\varepsilon^{(n)}$ can be thought of as being obtained by resampling each coordinate according to $\{1 - p_n, p_n\}$ with probability $1 - e^{-\varepsilon}$.

Whenever the dependency on n is clear, we will drop n in the superscript of $(X_t^{(n)})$.

The concept of noise sensitivity of sequences of Boolean functions was first defined in [1] as a measure of to what extent knowledge about $f_n(X_0)$ would help to predict $f_n(X_\varepsilon)$. Our definition is the same as the definition used in e.g. [2], and is equivalent to what is called being *asymptotically noise sensitivity* in [1].

Definition 1.1. A sequence of Boolean functions $f_n : \{-1, 1\}^n \to \{-1, 1\}$ is said to be *noise sensitive* with respect to (p_n) if for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathrm{Cov}_{p_n}[f_n(X_0), f_n(X_\varepsilon)] = 0.$$

In the same paper, the authors also introduced the concept of noise stability, which captures a possible opposite behaviour.

Definition 1.2. A sequence of Boolean functions $f_n : \{-1, 1\}^n \to \{-1, 1\}$ is said to be *noise stable* with respect to (p_n) if

$$\lim_{\varepsilon\to 0} \sup_n P_{p_n}[f_n(X_\varepsilon) \neq f_n(X_0)] = 0. \tag{1}$$

Note that as $\{-1, 1\}^n$ is finite for every $n \in \mathbb{N}$, (1) is equivalent to

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} P_{p_n}[f_n(X_\varepsilon) \neq f_n(X_0)] = 0.$$

When using these definitions, one generally assumes that the sequence (f_n) of Boolean functions is *nondegenerate*, meaning that

$$-1 < \liminf_{n\to\infty} \mathbb{E}[f_n(X_0)] \leq \limsup_{n\to\infty} \mathbb{E}[f_n(X_0)] < 1.$$

It is easy to show that if (f_n) is not nondegenerate, then it is both noise sensitive and noise stable.

In [3], another measure of the stability of a sequence of Boolean functions was introduced. One motivation was that the two definitions above, although giving information about $f_n(X_t)$ at two distinct times $t = 0$ and $t = \varepsilon$, gives no information about $f_n(X_t)$ for intermediate times t.