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# Extensions of the sewing lemma with applications

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**Abstract**

We give several extensions of the sewing lemma of Feyel and de La Pradelle and show how these results generalize Young's integration theory in a simple and natural way. For illustrative purposes, we apply the lemma to integrals involving discontinuous functions of a fractional Brownian motion with the Hurst index $H > 1/2$.


*Keywords:* The sewing lemma; Riemann–Stieltjes integrals; Stochastic integrals; Fractional Brownian motion

## 1. Introduction

The paper contributes to Young's integration theory. We study conditions under which the integral $\int_a^b Y\,dX$ exists as a limit of forward or ordinary Riemann–Stieltjes sums. We also discuss the existence of two-dimensional integrals of the form $\int_a^b \int_c^d Y\,dX$.

Our basic tool is the famous sewing lemma of Feyel and de La Pradelle [6] that is very useful in rough path theory introduced by Lyons. The existing versions of the lemma from [6] and [7] do not cover certain cases of interest discussed below. In this paper, we derive a general version of this lemma that can be further extended to the multidimensional setting.

The sewing lemma is intimately related to Young's results [18,19] concerning Riemann–Stieltjes integrals. In [19], Young shows that the condition

$$\sum_{n=1}^{\infty} \varphi^{-1}(n^{-1})\psi^{-1}(n^{-1}) < \infty \text{ or, equivalently, } \int_0^1 \varphi^{-1}(u)\psi^{-1}(u)\frac{du}{u^2} < \infty \qquad (1)$$

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implies that the Riemann–Stieltjes integral $\int_a^b f dg$ exists for any functions $f, g$ of bounded $\varphi$-and $\psi$-variations on $[a, b]$, respectively, with no common discontinuities. Here $\varphi, \psi \in \Phi$ with $\Phi$ being the class of continuous strictly increasing functions $\phi$ on $\mathbb{R}_+$ with $\phi(\mathbb{R}_+) = \mathbb{R}_+$, and the $\phi$-variation of a function $f$ on $[a, b]$ is defined by

$$\mathcal{V}_\phi(f, [a, b]) = \sup \sum_{i=1}^{n} \phi(|f(t_i) - f(t_{i-1})|),$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_n = b$. If $\varphi(x) = x^p$ and $\psi(x) = x^q$ for some $p, q \geqslant 1$, we get $p$- and $q$- variations, respectively, and (1) holds whenever $1/p + 1/q > 1$.

Dudley and Norvaiša [5] provide an extensive list of results concerning the $\phi$-variation of stochastic processes and describe different applications of Young's and related results in probability and statistics (see Chapter 12). Towghi [16] derives multidimensional extensions of Young's results [18]. In [13], Ruzmaikina studies Riemann–Stieltjes integrals with Hölder continuous functions, a special case covered by Young's results, and uses upper bounds on such integrals to prove the existence and uniqueness of solutions of ordinary differential equations with Hölder continuous forcing. Related problems are also studied by Lyons and many other authors in the context of rough path theory (e.g., see [9] and [10] among others).

Young's integration theory does not cover certain cases of interest in stochastic calculus, including some integrals involving a fractional Brownian motion $B_H$ with the Hurst index $H > 1/2$. The most prominent example is the integral $\int_0^1 I(B_H > 0) dB_H$ that arises in non-semimartingale models of the stock market (see [1]). It is hard to check its existence by referring to (1), since, by the self-similarity of $B_H$, the indicator process $X = I(B_H > 0)$ has unbounded $\phi$-variation on $[0, 1]$ for any $\phi \in \Phi$ in any reasonable sense (see Proposition 3.5). This fact has motivated a number of papers [1,3,15] employing generalized Lebesgue–Stieltjes integrals defined via Riemann–Liouville fractional derivatives (see [12,20]). However, it is shown in [1,3,15] that these integrals coincide with Riemann–Stieltjes integrals under certain assumptions.

Our purpose is to extend the sewing lemma and Young's condition (1) in a simple and natural way to cover the cases of interest involving the fractional Brownian motion. Our arguments are close to those from ordinary calculus as in [6,7,16], and Chapter 3 of [5]. In particular, we do not use fractional derivatives.

The paper is structured as follows. The sewing lemma is given in Section 2. Section 3 deals with applications. Section 4 contains a two-dimensional extension of the sewing lemma. The proofs are deferred to Section 5 and an Appendix.

## 2. The sewing lemma

This section contains a general version of the sewing lemma from [6]. To state it, we need to introduce some notation.

Let $\Delta(a, b) = \{(s, t) : a \leqslant s \leqslant t \leqslant b\}$ for $a \leqslant b$. For given $T > 0$, we say that $f : \Delta(0, T) \to \mathbb{R}_+$ is a control function if $f$ is non-decreasing in the sense that $f(s, t) \leqslant f(a, b)$ whenever $[s, t] \subseteq [a, b]$. Let us also introduce the following quantity

$$I_f(a, b) = \iint_{\Delta(a,b)} \frac{f(s, t)}{(t - s)^2} ds\, dt + \int_a^b \frac{f(a, t)}{t - a} dt + \int_a^b \frac{f(s, b)}{b - s} ds + f(a, b)$$

for $a \leqslant b$.

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