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Existentially generated subfields of large fields

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ABSTRACT

We study subfields of large fields which are generated by infinite existentially definable subsets. We say that such subfields are **existentially generated**.

Let L be a large field of characteristic exponent p, and let $E \subseteq L$ be an infinite existentially generated subfield. We show that E contains $L^{(p^n)}$, the p^n-th powers in L, for some $n < \omega$. This generalises a result of Fehm from [4], which shows $E = L$, under the assumption that L is perfect. Our method is to first study existentially generated subfields of henselian fields. Since L is existentially closed in the henselian field $L((t))$, our result follows.

Large fields were introduced in [9] by Pop: A field L is **large**[1] if every smooth curve defined over L with at least one L-rational point has infinitely many L-rational points. A survey of the theory of large fields is given in [2].

Our fields have characteristic exponent p, i.e. p is the characteristic, if this is positive, and otherwise $p = 1$. A subset $X \subseteq L$ is **existentially definable** if it is defined by an existential formula from the language of rings, allowing parameters. We denote by (X)

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[1] Large fields are also known as *ample fields*.

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the subfield generated by X. A subfield $E \subseteq L$ is **existentially generated** if there is an infinite existentially definable subset $X \subseteq L$ which generates E, i.e. $E = (X)$.

In section 5 we prove the following theorem.

Theorem 1. *Let L be a large field of characteristic exponent p, and let $E \subseteq L$ be an existentially generated subfield. Then we have*

$$L^{(p^n)} \subseteq E,$$

for some $n < \omega$, where $L^{(p^n)} = \{x^{p^n} | x \in L\}$ is the subfield of p^n-th powers.

The motivation for this work was the following result of Arno Fehm.

Theorem 2 *(Corollary 9, [4]). A perfect large field L has no existentially L-definable proper infinite subfields.*

In fact, using our terminology, Fehm's method immediately shows that *a perfect large field L has no existentially generated proper subfields*. For imperfect L and each $n < \omega$, the subfield $L^{(p^n)}$ is existentially definable, without parameters, by using the Frobenius map. Moreover, if we use parameters then we are able to existentially define various extensions of the subfields $L^{(p^n)}$. Thus, our result generalises Theorem 2 by removing the assumption that L is perfect. On the other hand, if the characteristic of L is zero, then L is necessarily perfect, so Fehm's result already applies.

The key to our method is to study the same problem in a henselian field K, i.e. a field equipped with a nontrivial henselian valuation. First, we recall some facts about separable field extensions in section 1. Then in the context of an arbitrary field, in section 2 we introduce and study '*big* subfields'; and in section 3 we introduce and study '*uniformly big* subfields'. In section 4 we show that existentially generated subfields of henselian fields are uniformly big, and that they contain 'sufficiently many' points of $K^{(p^\infty)}$. From this we can deduce Theorem 1, restricted to henselian fields. Finally, in section 5, we use the fact that L is existentially closed in the henselian field $L((t))$ to finish the proof of Theorem 1.

Notation. Throughout, C, E, F, K, L will denote fields, C will usually be a subfield 'of parameters', K will be henselian, and L will be large. To avoid confusion between Cartesian products and sets of powers, for $n < \omega$ and a set X, we let $X^n = X \times \ldots \times X$ denote the n-fold Cartesian product, and let $X^{(n)} = \{x^n | x \in X\}$ denote the set of n-th powers of elements from X. Sometimes it will be convenient to think of tuples as being indexed by a tuple of variables. If $\mathbf{x} = (x_1, ..., x_m)$ is an m-tuple of variables, we write $X^{\mathbf{x}}$ for the set of m-tuples from X indexed by $\mathbf{x}$.

Let $\mathbf{x} = (x_1, ..., x_m)$, $\mathbf{y} = (y_1, ..., y_n)$ be two tuples of variables. Despite the abuse of language, we say $\mathbf{x}$ is a **subtuple** of $\mathbf{y}$ if $\{x_1, ..., x_m\} \subseteq \{y_1, ..., y_n\}$. In this case, we write $\mathrm{pr}_{\mathbf{x}} : X^{\mathbf{y}} \longrightarrow X^{\mathbf{x}}$ for the **projection** that maps each $\mathbf{y}$-tuple to its subtuple corresponding to $\mathbf{x}$.