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Gilman's conjecture ☆

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ABSTRACT

We prove a conjecture made by Gilman in 1984 that the groups presented by finite, monadic, confluent rewriting systems are precisely the free products of free and finite groups.

Contents

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1. Introduction

Many algebraic structures are defined by, or at least naturally accompanied by, a finite rewriting system. A *rewriting system* is a pair (Σ, T), where Σ is a finite alphabet of symbols, Σ^* denotes the set of all words over the alphabet Σ, and $T \subset \Sigma^* \times \Sigma^*$ is a set of rewriting rules. Each rewriting rule (L, R) specifies an allowable replacement: whenever L appears as a subword, it may be replaced by R. We write $U \xrightarrow{*} V$, if the word U can be transformed into the word V by application of a finite sequence of rewriting rules. The reflexive and symmetric closure of $\xrightarrow{*}$ is an equivalence relation on Σ^* whose equivalence classes form a monoid under the operation of concatenation of representatives. Sometimes this monoid is a group.

A fundamental question of combinatorial group theory and the foundations of computer science asks which algebraic classes of groups can be characterized by the types of rewriting systems presenting groups in that class. Having a nice rewriting system for a particular group often allows one to perform efficient computations in the group—for example, solving the word or conjugacy problems. A substantial effort, with contributions from many authors spanning a period of more than three decades ([5], [9], [2], [3], [1], [6], [11], [15], [8], [14], and more), has been made in pursuit of a complete algebraic characterization of groups presented by *length-reducing* rewriting systems (those in which each application of a rewriting rule shortens a word). A summary of many results in this program can be found in [11]; we mention a few relevant results here.

One can strengthen the requirement that (Σ, T) is length-reducing in various ways, restricting attention to *monadic*, *2-monadic*, or *special* rewriting systems. (See Section 2.2 for precise definitions.) It is common to consider *confluent* rewriting systems, but this can be relaxed to require only that a rewriting system is *confluent on* [1], the equivalence class of the empty word (see, for example, [8], [15]).

Cochet [5] proved that a group G is presented by a finite, special, confluent rewriting system if and only if G is the free product of finitely many cyclic groups. Diekert [6] showed that every group presented by a finite, monadic, confluent rewriting system is virtually free. If, in addition, the rewriting system is *inverse-closed* (every element represented by a generator has an inverse which is represented by a generator), then Avenhaus and Madlener [2] showed that (Σ, T) must present a *plain group*, that is, a free product of a finitely generated free group with finitely many finite groups. Gilman [9] conjectured in 1984 that this was the case even without assuming that (Σ, T) is inverse-closed. Avenhaus, Madlener and Otto [3] proved Gilman's conjecture in the special case that in each rewriting rule the left-hand side has length exactly two. The second author proved Gilman's conjecture in the special case that every generator has finite order [14]. Our main result resolves Gilman's conjecture in its full generality:

Theorem 5.3. *A group G is presented by a finite, monadic, confluent rewriting system (Σ, T) if and only if G is a plain group.*