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Two extremal problems on intersecting families

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ABSTRACT

In this short note, we address two problems in extremal set theory regarding intersecting families. The first problem is a question posed by Kupavskii: is it true that given two disjoint cross-intersecting families $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$, they must satisfy $\min\{|\mathcal{A}|, |\mathcal{B}|\} \le \frac{1}{2}\binom{n-1}{k-1}$? We give an affirmative answer for $n \ge 2k^2$, and construct families showing that this range is essentially the best one could hope for, up to a constant factor. The second problem is a conjecture of Frankl. It states that for $n \ge 3k$, the maximum diversity of an intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ is equal to $\binom{n-3}{k-2}$. We are able to find a construction beating the conjectured bound for n slightly larger than $3k$, which also disproves a conjecture of Kupavskii.

1. Disjoint cross-intersecting families

One of the most famous results in extremal set theory is the Erdős–Ko–Rado Theorem [2]: for $n \ge 2k$, an intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ has size at most $\binom{n-1}{k-1}$. The Erdős–Ko–Rado Theorem has many analogues and generalizations. One particularly interesting generalization is by considering two families instead of one. We say two families $\mathcal{A}$ and $\mathcal{B}$ are *cross-intersecting*, if for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $A \cap B \neq \emptyset$. Pyber [16] showed that when n is large in k, l, for $\mathcal{A} \subset \binom{[n]}{k}$, $\mathcal{B} \subset \binom{[n]}{l}$, we have $|\mathcal{A}||\mathcal{B}| \le \binom{n-1}{k-1}\binom{n-1}{l-1}$. Later the same inequality for a precise range $n \ge 2\max\{k, l\}$ was established by Matsumoto and Tokushige [15]. The Erdős–Ko–Rado Theorem follows immediately by setting $k = l$ and $\mathcal{A} = \mathcal{B}$.

Recently Kupavskii [12] asked the following question: given two cross-intersecting families $\mathcal{A}$ and $\mathcal{B}$ that are disjoint, is it true that

$$\min\{|\mathcal{A}|, |\mathcal{B}|\} \le \frac{1}{2}\binom{n-1}{k-1}?$$

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This bound, if true, is clearly tight. This is because we can always split the extremal example in Erdős–Ko–Rado Theorem, i.e. a 1-star $\mathcal{S}$, into two subfamilies $\mathcal{S}_1, \mathcal{S}_2$ as evenly as possible. Then $\mathcal{A} = \mathcal{S}_1$ and $\mathcal{B} = \mathcal{S}_2$ are cross-intersecting and disjoint, and each has about $\frac{1}{2}\binom{n-1}{k-1}$ subsets. In this section, we give a positive answer to this question for n quadratic in k.

Theorem 1.1. *For integers $k \geq 2$ and $n \geq 2k^2$, given two disjoint cross-intersecting families $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$, we have*

$$\min\{|\mathcal{A}|, |\mathcal{B}|\} \leq \frac{1}{2}\binom{n-1}{k-1}.$$

As a warm-up, first we show that when n is at least cubic in k, this statement is true. Consider a pair of disjoint crossing-intersecting families $\mathcal{A}$ and $\mathcal{B}$ of k-sets of $[n]$. If both $\mathcal{A}$ and $\mathcal{B}$ are intersecting, then $\mathcal{A} \cup \mathcal{B}$ is also intersecting, by the Erdős–Ko–Rado Theorem, for $n \geq 2k$, we have

$$|\mathcal{A}| + |\mathcal{B}| \leq \binom{n-1}{k-1},$$

and thus we have the desired inequality

$$\min\{|\mathcal{A}|, |\mathcal{B}|\} \leq \frac{1}{2}\binom{n-1}{k-1}.$$

Now suppose at least one of $\mathcal{A}$ and $\mathcal{B}$ is not intersecting, without loss of generality we may assume that $\mathcal{A}$ is not intersecting, then there exists $A_1, A_2 \in \mathcal{A}$, such that $A_1 \cap A_2 = \emptyset$. Now the number of sets that intersect with both A_1 and A_2 provides an upper bound for $|\mathcal{B}|$, which is at most

$$k^2\binom{n-2}{k-2} = \frac{k^2(k-1)}{(n-1)}\binom{n-1}{k-1} < \frac{1}{2}\binom{n-1}{k-1}$$

when $n \geq 2k^3$.

Next we will improve the range to $n \geq 2k^2$. The main tool used in this proof is the technique of shifting, which allows us to limit our attention to sets with certain structure. In this section we will only state and prove some relevant results. For more background on the applications of shifting in extremal set theory, we refer the reader to the survey [3] by Frankl.

Proof of Theorem 1.1. For $k = 2$, the families $\mathcal{A}$ and $\mathcal{B}$ can be viewed as two n-vertex graphs. When $n \geq 2k^2 = 8$, $\frac{1}{2}\binom{n-1}{k-1} \geq 3$. Assume $|\mathcal{A}|, |\mathcal{B}| \geq 4$. If they are both intersecting, then the previous discussion gives the conclusion. Otherwise suppose $\mathcal{A}$ is not intersecting, then it contains two disjoint edges e and f. $\mathcal{B}$ can only contain at most 4 edges intersecting both e and f. However if all four edges are present, then $|\mathcal{A}| \leq 2$. This settles the $k = 2$ case.

Now we may assume that $k \geq 3$ and $n \geq 2k^2$, suppose there exist disjoint cross-intersecting families $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ such that

$$\min\{|\mathcal{A}|, |\mathcal{B}|\} > \frac{1}{2}\binom{n-1}{k-1}.$$

We will prove the following statement: given positive integers $k, l \geq 3$, and $n \geq k + l$, suppose $\mathcal{A} \subset \binom{[n]}{k}$, $\mathcal{B} \subset \binom{[n]}{l}$ are cross-intersecting. If $|\mathcal{A}| > \max\{k, l\}\binom{n-2}{k-2}$ and $|\mathcal{B}| > \max\{k, l\}\binom{n-2}{l-2}$, then there exists some element x contained in every subset of $\mathcal{A}$ and $\mathcal{B}$, i.e., $\mathcal{A} \cup \mathcal{B}$ is *trivial*. Assuming this claim, if there exists x such that $\mathcal{A}, \mathcal{B}$ are subfamilies of the 1-star centered at x, then $|\mathcal{A} \cup \mathcal{B}| \leq \binom{n-1}{k-1}$, and Theorem 1.1 follows from the disjointness of $\mathcal{A}$ and $\mathcal{B}$. Otherwise, either $|\mathcal{A}|$ or $|\mathcal{B}|$ has to be strictly smaller than

$$k\binom{n-2}{k-2} = k \cdot \frac{k-1}{n-1}\binom{n-1}{k-1} \leq \frac{1}{2}\binom{n-1}{k-1}$$

for $n \geq 2k^2$, which also proves Theorem 1.1.

Assume $k \leq l$. We prove by contradiction, suppose $\mathcal{A}$ and $\mathcal{B}$ satisfy the size condition, and $\mathcal{A} \cup \mathcal{B}$ is non-trivial, then both $\mathcal{A}$ and $\mathcal{B}$ must be non-trivial. Otherwise suppose $\mathcal{A}$ is trivial with center a,