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The two-level finite difference schemes for the heat equation with nonlocal initial condition

Jesús Martín-Vaquero[a,*], Svajūnas Sajavičius[b]

[a] Department of Applied Mathematics, Institute of Fundamental Physics and Mathematics, Universidad de Salamanca, Salamanca, Spain
[b] Department of Software Engineering, Kaunas University of Technology, Student\k{u} str. 50, LT-51368 Kaunas, Lithuania

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ABSTRACT

In this paper, the two-level finite difference schemes for the one-dimensional heat equation with a nonlocal initial condition are analyzed. As the main result, we obtain conditions for the numerical stability of the schemes. In addition, we revise the stability conditions obtained in [21] for the Crank–Nicolson scheme. We present several numerical examples that confirm the theoretical results within linear, as well as nonlinear problems. In some particular cases, it is shown that for small regions of the time step size values, the explicit FTCS scheme is stable while certain implicit methods, such as Crank–Nicolson scheme, are not.

1. Introduction

Mathematical models arising in various fields of science and engineering very often are expressed in terms of partial differential equations (PDEs) with nonlocal initial or boundary conditions. For example, we can mention models arising in thermoelasticity [1], thermodynamics [2], geology [3], hydrodynamics [4], biological fluid dynamics [5] or plasma physics [6]. The present paper is focused on differential problems with *nonlocal initial conditions*. Such problems generalize the classical or time-periodic problems and can be seen as the control problems with initial conditions.

Nonclassical problems with nonlocal initial conditions are important because of their practical applications in modeling and investigation of sewage caused pollution processes in rivers and seas. Such problems are also used when investigating radionuclides propagation in Stokes fluid, diffusion and flow in porous media [7–9]. Nonlocal initial conditions also arise in mathematical models applied in meteorology since the use of time-averaged data instead of the initial data only leads to more reliable long-term weather forecasts [10].

In this paper, as a model problem we consider the one-dimensional parabolic (heat) equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{1}$$

subject to homogeneous boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{2}$$

* Corresponding author.
E-mail addresses: jesmarva@usal.es, jesusmpapers@outlook.es (J. Martín-Vaquero), svajunas.sajavicius@ktu.lt (S. Sajavičius).
URL: http://web.usal.es/~jesmarva (J. Martín-Vaquero), http://orcid.org/0000-0002-1384-1365 (S. Sajavičius)

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and nonlocal discrete–integral initial condition

$$u(x,0)=\sum_{j=1}^{m}\alpha_j u(x,T_j)+\int_0^T \upsilon(\tau)u(x,\tau)\mathrm{d}\tau+\varphi(x),\quad x\in\Omega. \tag{3}$$

Here $\Omega=(0,L)$ is a spatial interval, $\alpha_j\neq 0$, $0<T_j\leq T$ $(1\leq j\leq m)$, $\upsilon\in L^1(0,T)$.

Nonlocal in time problems for parabolic equations were considered in [11] and, later, in [7–9,12–14] (see also references therein). The solvability of various differential problems with nonlocal initial conditions systematically has been studied in papers [15–18].

The existence and uniqueness results related to the problem (1)–(3) are given in paper [17]. If $\varphi\in L^2(\Omega)$, $f\in L^2([0,T];H^{-1}(\Omega))$ and

$$1-\sum_{j=1}^{m}\frac{\alpha_j+|\alpha_j|}{2}\geq\int_0^T|\upsilon(\tau)|\mathrm{d}\tau,$$

then the problem (1)–(3) has a unique solution $u\in C^0([0,T];L^2(\Omega))\cap L^2([0,T];H_0^1(\Omega))$.

In recent decades, numerical methods for the solution of PDEs with nonlocal boundary conditions are developed and investigated very actively. However, only a few studies are related to the numerical solution of PDEs with nonlocal initial conditions. For example, the error estimates for the semidiscrete finite element approximation of the solution to linear parabolic equation have been obtained in paper [19]. Iterative finite element approximations of the solutions to parabolic equations with certain nonlocal initial conditions have been studied in [20]. For the numerical solution of nonlinear parabolic problems with a nonlocal initial condition, iterative finite difference schemes have been proposed and analyzed in [21,22]. In papers [23,24], the finite difference schemes for the one-dimensional parabolic (heat) equation with nonlocal discrete initial conditions were examined. For the solution of this problem, a polynomial-based collocation technique has been suggested in paper [25].

For the numerical solution of nonlinear parabolic problems with a nonlocal initial condition, and iterative finite difference scheme has been investigated in papers [21,22]. In [21], the stability and convergence of several finite difference schemes have been studied.

In this paper, we extend the results presented in paper [21] to a more general class of methods, including numerical schemes which were applied previously without studying their stability properties. Additionally, we revise Theorem 3.2 proved in the paper [21] by adding a new constraint for the time step size. The revised analysis leads to the stability conditions which were not considered in [21,23]. The numerical results presented in this paper show that conditions provided in [21,23] cannot guarantee the stability of the corresponding numerical schemes. Also, we demonstrate that in some particular cases the forward-time central-space (FTCS) explicit numerical scheme is stable, while some implicit methods (such as Crank–Nicolson scheme) are not. From the point of view of the classical theory of finite difference schemes, this is a quite surprising result. The examined methods can be naturally extended for nonlinear problems. For illustration, we present the results of a numerical experiment with a nonlinear problem.

The paper is organized as follows. The two-level finite difference schemes for the solution of the considered nonclassical problem are presented in Section 2. In Section 3, we analyze these schemes by studying their stability and accuracy properties. To verify the theoretical results and demonstrate the efficiency of the methods, several numerical experiments have been conducted. The results of these experiments are presented in Section 4. Finally, some remarks in Section 5 conclude the paper.

2. Two-level finite difference schemes

For the numerical solution of the considered problem (1)–(2) we apply the finite difference technique [26–29]. We construct a family of finite difference schemes depending on several parameters.

The problem domain $\Omega\times[0,T]$ is discretized by the uniformly distributed grid points (x_i, t_n), where

$$\begin{aligned}&x_i=ih,\quad i=0,1,\dots,N,\quad Nh=L,\\&t_n=n\tau,\quad n=0,1,\dots,M,\quad M\tau=T,\end{aligned}$$

where h and τ are space and time step sizes. We assume that $\{T_1,T_2,\dots,T_m\}\subset\{t_0,t_1,\dots,t_M\}$ and $T_j=t_{n_j}$ for some $n_j\in\{0,1,\dots,M\}$.

The one-dimensional parabolic equation (1) is approximated by the following finite difference equations:

$$\begin{aligned}\frac{u_i^{n+1}-u_i^n}{\tau}&=\sigma\frac{u_{i-1}^{n+1}-2u_i^{n+1}+u_{i+1}^{n+1}}{h^2}+(1-\sigma)\frac{u_{i-1}^{n}-2u_i^{n}+u_{i+1}^{n}}{h^2}\\&+c_{00}f_{i-1}^{n+1}+c_{01}f_i^{n+1}+c_{00}f_{i+1}^{n+1}+c_{10}f_{i-1}^{n}+c_{11}f_i^{n}+c_{10}f_{i+1}^{n},\end{aligned}\tag{4}$$

where σ is the weight of the scheme $(0\leq\sigma\leq 1)$, and c_{00}, c_{01}, c_{10}, c_{11} are coefficients to be determined later (see Section 3.2). Depending on the values of σ, we distinguish several special cases:

- $\sigma=1$: the backward-time central-space (BTCS) scheme;