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Developing high order methods for the solution of systems of nonlinear equations

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ABSTRACT

Two families of order six for the solution of systems of nonlinear equations are developed and compared to existing schemes of order up to six. We have found that one of the methods in the literature has been rediscovered. The comparison is based on the total cost of an iteration and the performance on 14 examples of systems of dimensions 2–9.

1. Introduction

The solution of systems of nonlinear equations is required whenever a nonlinear partial differential equation is approximated. The most well known scheme is Newton's method given by (see e.g. [1,2] or [3])

$$x_{n+1} = x_n - \left[F'(x_n)\right]^{-1} F(x_n), \tag{1}$$

where $F(x) = 0$ is the system to be solved and $F'(x_n)$ is the Jacobian. Assuming one has a close enough initial vector x_0 and that the Jacobian never vanishes for any iterate x_n, the method will converge quadratically. This method requires the construction of the Jacobian and the solution of a system of linear equation at every step. To reduce the cost, one can keep the Jacobian fixed for say k iterates. In this case the order is $k+1$, e.g. if we keep the Jacobian for 3 iterates, we get a fourth order method. This is called modified Newton's method, denoted by MN, and given by

$$\begin{aligned} y_n &= x_n - \left[F'(x_n)\right]^{-1} F(x_n), \\ z_n &= y_n - \left[F'(x_n)\right]^{-1} F(y_n), \\ x_{n+1} &= z_n - \left[F'(x_n)\right]^{-1} F(z_n). \end{aligned} \tag{2}$$

There are other ways to modify the procedure, e.g. Steffensen method using divided difference to replace the Jacobian, see e.g. [4], Ezquerro et al. [5] and also a survey by Rheinboldt [6]. Artidiello et al. [7] have suggested the use of divided difference instead of one of the Jacobians.

Neta [8] has developed a fourth order method, denoted Neta4, based on his sixth order method for the solution of a single equation [9]. The method is given by

$$y_n = x_n - \left[F'(x_n)\right]^{-1} F(x_n),$$

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$$
\begin{aligned}
z_n &= y_n - Q_1(x_n, y_n)\big[F'(x_n)\big]^{-1}F(y_n),\\
x_{n+1} &= z_n - Q_2(x_n, y_n)\big[F'(x_n)\big]^{-1}F(z_n),
\end{aligned}
\tag{3}
$$

where the weight functions chosen here are

$$
Q_1(x_n, y_n) = \frac{F^T(x_n)F(x_n) + 2F^T(x_n)F(y_n) - a(a-2)F^T(y_n)F(y_n)}{F^T(x_n)F(x_n) - (a-2)^2F^T(y_n)F(y_n)},
\tag{4}
$$

and

$$
Q_2(x_n, y_n) = \frac{F^T(x_n)F(x_n) + 2F^T(x_n)F(y_n) - 3F^T(y_n)F(y_n)}{F^T(x_n)F(x_n) - 9F^T(y_n)F(y_n)},
\tag{5}
$$

and the parameter a was chosen as zero. The original idea is to have the weight function chosen in such a way that the method will be of higher order than 4. This was not successful as the numerical experiments will show.

Methods of higher order than 4 were developed in the literature and we will quote several methods of order five and six. Cordero et al. [10] have developed a fifth order method, denoted here by CHMT, given by

$$
\begin{aligned}
y_n &= x_n - \big[F'(x_n)\big]^{-1}F(x_n),\\
z_n &= x_n - 2\big[F'(x_n) + F'(y_n)\big]^{-1}F(x_n),\\
x_{n+1} &= z_n - \big[F'(y_n)\big]^{-1}F(z_n).
\end{aligned}
\tag{6}
$$

Another fifth order family of methods due to Sharma et al. [11] is given by

$$
\begin{aligned}
y_n &= x_n - \theta\big[F'(x_n)\big]^{-1}F(x_n),\\
z_n &= x_n - \left[\left(1 + \frac{1}{2\theta}\right)I - \frac{1}{2\theta}\big[F'(x_n)\big]^{-1}F'(y_n)\right]\big[F'(x_n)\big]^{-1}F(x_n),\\
x_{n+1} &= z_n - \left[\left(1 + \frac{1}{\theta}\right)I - \frac{1}{\theta}\big[F'(x_n)\big]^{-1}F'(y_n)\right]\big[F'(x_n)\big]^{-1}F(z_n).
\end{aligned}
\tag{7}
$$

The case $\theta = 1$ was shown to be the best and we will use that here and denote it SSK. We also used $\theta = 2/3$ to match with the other schemes by [12,13].

The first family of methods of order six is found in Hueso et al. [12]

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}\big[F'(x_n)\big]^{-1}F(x_n),\\
z_n &= x_n - \left[\frac{5-8a_2}{8}I + a_2\big[F'(y_n)\big]^{-1}F'(x_n) + \frac{a_2}{3}\big[F'(x_n)\big]^{-1}F'(y_n)\right.\\
&\quad \left. + \frac{9-8a_2}{24}\left(\big[F'(y_n)\big]^{-1}F'(x_n)\right)^2\right]\big[F'(x_n)\big]^{-1}F(x_n),\\
x_{n+1} &= z_n - \left[b_1 I - \frac{3+8b_1}{8}\big[F'(y_n)\big]^{-1}F'(x_n) + \frac{15-8b_1}{24}\big[F'(x_n)\big]^{-1}F'(y_n)\right.\\
&\quad \left. + \frac{9+4b_1}{12}\left(\big[F'(y_n)\big]^{-1}F'(x_n)\right)^2\right]\big[F'(y_n)\big]^{-1}F(z_n).
\end{aligned}
\tag{8}
$$

Two members were experimented with in [12] and chosen because of their computational efficiency. These are

- HMT1, when $a_2 = 9/8$ and $b_1 = -9/4$

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}\big[F'(x_n)\big]^{-1}F(x_n),\\
z_n &= x_n - \left[-\frac{1}{2}I + \frac{9}{8}\big[F'(y_n)\big]^{-1}F'(x_n) + \frac{3}{8}\big[F'(x_n)\big]^{-1}F'(y_n)\right]\big[F'(x_n)\big]^{-1}F(x_n),\\
x_{n+1} &= z_n - \left[-\frac{9}{4}I + \frac{15}{8}\big[F'(y_n)\big]^{-1}F'(x_n) + \frac{11}{8}\big[F'(x_n)\big]^{-1}F'(y_n)\right]\big[F'(y_n)\big]^{-1}F(z_n).
\end{aligned}
\tag{9}
$$

- HMT2, when $a_2 = 0$ and $b_1 = -9/4$

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}\big[F'(x_n)\big]^{-1}F(x_n),\\
z_n &= x_n - \left[\frac{5}{8}I + \frac{3}{8}\left(\big[F'(y_n)\big]^{-1}F'(x_n)\right)^2\right]\big[F'(x_n)\big]^{-1}F(x_n),\\
x_{n+1} &= z_n - \left[-\frac{9}{4}I + \frac{15}{8}\big[F'(y_n)\big]^{-1}F'(x_n) + \frac{11}{8}\big[F'(x_n)\big]^{-1}F'(y_n)\right]\big[F'(y_n)\big]^{-1}F(z_n).
\end{aligned}
\tag{10}
$$