Journal of Statistical Planning and Inference xx (xxxx) xxx–xxx

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Tripling of fractional factorial designs

Zujun Ou [a], Minghui Zhang [a], Hong Qin [b,*]

[a] *College of Mathematics and Statistics, Jishou University, Jishou 416000, China*
[b] *College of Mathematics and Statistics, Central China Normal University, Wuhan 430079, China*

A R T I C L E I N F O

Article history:
Received 19 October 2017
Received in revised form 4 March 2018
Accepted 11 June 2018
Available online xxxx

MSC:
62K15
62K99

Keywords:
Fractional factorial design
Generalized minimum aberration
Level permutation
Tripling
Uniform design

A B S T R A C T

Doubling and level permutation of factors are widely used in the construction of uniform designs and minimum aberration designs. Based on the viewpoint that double design is the orthogonal combination of all possible level permutations of its initial two-level design, a method of tripling for three-level design, which triples both the run size and number of factors of initial design, is proposed by orthogonally combining all possible level permutations of its initial design in this paper. The wrap-around L_2-discrepancy of triple design is expressed by the wordlength pattern of its initial design, and a tight lower bound of the wrap-around L_2-discrepancy of triple design is obtained. An efficient method for constructing uniform minimum aberration designs is proposed based on the projection of triple design. These constructed designs have better properties, such as minimum aberration and lower discrepancy, than existing uniform designs, and are recommended for use in practice.

1. Introduction

Fractional factorial designs are widely used in various area such as manufacturing, pharmaceutics, sciences and engineering. They are often chosen by the minimum aberration criterion (Fries and Hunter, 1980) and its extension, generalized minimum aberration criterion (Xu and Wu, 2001) and minimum generalized criterion (Ma and Fang, 2001).

Chen and Cheng (2006) discussed the method of doubling for constructing two-level fractional factorial designs, in particular, those of resolution IV. Suppose that X is an $n \times s$ matrix with two distinct entries, 1 and −1. Then the double of X is the $2n \times 2s$ matrix $D(X) = \begin{pmatrix} X & X \\ X & -X \end{pmatrix}$. Doubling plays an important role in the construction of maximal designs. Xu and Cheng (2008) developed a general complementary design theory for doubling. One can refer to Hu and Zhang (2009) and Ou and Qin (2010, 2017) for more details about doubling. From the definition of double design $D(X)$, both $-X$ and X can be regarded as the level permutation of X, and $D(X)$ is just the orthogonal combination of all possible level permutations of X. We may wonder whether the doubling process by orthogonal combination of all possible level permutations of two-level design could be extended to three-level design.

For designs with more than two levels, level permutation of one or more factors can alter their geometrical structures and statistical properties (Cheng and Ye, 2004). Recently, by considering all possible level permutations, Tang et al. (2012) proposed a novel construction method for three-level uniform designs under uniformity criterion measured by the centered L_2-discrepancy, Tang and Xu (2013) and Xu et al. (2014) extended their results to multi-level designs under uniformity criterion in terms of the centered L_2-discrepancy and wrap-around L_2-discrepancy, respectively. Furthermore, in Zhou and Xu (2014), this idea has been generalized into fractional factorial designs with any number of levels and any discrepancy

* Corresponding author.
E-mail address: qinhong@mail.ccnu.edu.cn (H. Qin).

https://doi.org/10.1016/j.jspi.2018.06.002
0378-3758/© 2018 Published by Elsevier B.V.

defined by a reproducing kernel. Tang and Xu (2014) studied level permutations for regular fractional factorial designs in order to improve their efficiency for screening quantitative factors.

The current research is inspired by the idea of doubling in the sense of orthogonal combination of all possible level permutations of its initial design. A method of tripling for three-level design, which triples both the run size and number of factors of initial three-level design, is proposed by combining all possible level permutations of its initial design in this paper. The wrap-around L_2-discrepancy of triple design is expressed by the wordlength pattern of its initial design, and a tight lower bound of the wrap-around L_2-discrepancy of triple design is obtained. Therefore, triple designs of generalized minimum aberration designs tend to have good uniformity under the wrap-around L_2-discrepancy. Based on these theoretical results, an efficient method for constructing uniform minimum aberration designs (Tang et al., 2012) is proposed by the projection of triple design. These constructed designs have better properties, such as minimum aberration and lower discrepancy, than existing uniform designs, and are recommended for use in practice.

The paper is organized as follows. In Section 2, some notations and preliminaries are included. Section 3 gives the concept of triple design and the uniformity of triple design under the wrap-around L_2-discrepancy is studied, the wrap-around L_2-discrepancy of triple design is expressed by the generalized wordlength pattern of its initial design, and a tight lower bound of the wrap-around L_2-discrepancy of triple design is obtained. In Section 4, an efficient method for constructing uniform minimum aberration designs is proposed by the projection of triple design, numerical results show that the constructed designs have better properties, such as minimum aberration and lower discrepancy, than existing uniform designs. Finally, some conclusions are given in Section 5.

2. Notations and preliminaries

Consider a class of n runs and s three-level factors U-type fractional factorial designs, denoted as $\mathcal{U}(n;3^s)$. A design $\mathcal{F}$ in $\mathcal{U}(n;3^s)$ can be presented as an $n \times s$ matrix with entries 0, 1, 2, with each element occurring equally often in each column. An orthogonal array of strength t and size n with s constrains, denoted by $\mathrm{OA}(n, 3^s, t)$, is a factorial design of n runs and s three-level factors such that all the level-combinations for any t factors appear equally often. The design $\mathcal{F}$ in $\mathcal{U}(n;3^s)$, of course, is an orthogonal array of strength 1.

For any design $\mathcal{F} \in \mathcal{U}(n;3^s)$, define $E_i(\mathcal{F}) = |\{(a,b) : a, b \in \mathcal{F}, d_H(a,b) = i\}|/n$ for $i = 0, \ldots, s$, where $d_H(a,b)$ is the Hamming distance between two runs a and b of $\mathcal{F}$, namely, the number of places where they differ, and $|\Omega|$ is the cardinality of a set Ω. The vector $\{E_0(\mathcal{F}), \ldots, E_s(\mathcal{F})\}$ is called the distance distribution of $\mathcal{F}$ in the literature (MacWilliams and Sloane, 1977). Based on the distance distribution of $\mathcal{F}$, Ma and Fang (2001) defined the generalized wordlength pattern and generalized minimum aberration criterion as follows.

Definition 1. The generalized wordlength pattern of an n runs and s three-level factors fractional factorial design $\mathcal{F}$ is defined by $W(\mathcal{F}) = \{A_1(\mathcal{F}), \ldots, A_s(\mathcal{F})\}$, where

$$A_i(\mathcal{F}) = \frac{1}{n}\sum_{j=0}^{s} P_i(j;s)E_j(\mathcal{F}), \quad i = 1, \ldots, s,$$

and $P_i(j;s) = \sum_{r=0}^{i}(-1)^r 2^{i-r}\binom{j}{r}\binom{s-j}{i-r}$ is the Krawtchouk polynomial $\left(\binom{y}{z} = 0 \text{ for } y < z\right)$. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two designs. Let t be the smallest integer such that $A_t(\mathcal{G}_1) \neq A_t(\mathcal{G}_2)$ in their generalized wordlength patterns, then $\mathcal{G}_1$ is said to have less generalized aberration than $\mathcal{G}_2$ if $A_t(\mathcal{G}_1) < A_t(\mathcal{G}_2)$. A three-level design $\mathcal{F}$ has generalized minimum aberration if no other three-level design has less generalized aberration than it.

A run $\mathrm{x}_j = (x_{j1}, \ldots, x_{js})$ in design $\mathcal{F} \in \mathcal{U}(n;3^s)$ with entries $\{0, 1, 2\}$ can be mapped to $(u_{j1}, \ldots, u_{js})$, where $u_{jl} = (2x_{jl}+1)/6, j = 1, \ldots, n;\ l = 1, \ldots, s$. Uniform design aims to make experimental points uniformly scattered on the experimental domain. As a measure of uniformity, discrepancy plays a key role in uniform design. Hickernell (1998) used the tool of reproducing kernel Hilbert spaces to define several discrepancies. Among them, the wrap-around L_2-discrepancy is attractable and the most frequently used (Fang et al., 2003, 2005; Zhou and Ning, 2008). In this paper, we consider the wrap-around L_2-discrepancy to measure the uniformity of design $\mathcal{F} \in \mathcal{U}(n;3^s)$, due to Hickernell (1998), can be expressed in the following close form

$$[WD(\mathcal{F})]^2 = -\left(\frac{4}{3}\right)^s + \frac{1}{n^2}\sum_{i,j=1}^{n}\prod_{l=1}^{s}\left[\frac{3}{2} - |u_{il} - u_{jl}|(1 - |u_{il} - u_{jl}|)\right]. \tag{1}$$

3. Tripling of three-level design and its uniformity

In this section, based on the idea of doubling and level permutation, a new construction method of three-level design $\mathcal{F}$ named as triple design is proposed, which triples both the run size and number of factors of initial design.

Suppose that X is an $n \times s$ matrix with two distinct entries, 1 and -1. Then the double of X is the $2n \times 2s$ matrix $D(X) = \begin{pmatrix} X & X \\ X & -X \end{pmatrix}$, X can be regarded as the image of X by mapping $(-1, 1)$ to $(-1, 1)$, and $-X$ can be regarded as the

Please cite this article in press as: Ou Z., et al., Tripling of fractional factorial designs. J. Statist. Plann. Inference (2018), https://doi.org/10.1016/j.jspi.2018.06.002.