Engineering Structures 177 (2018) 331–344

Contents lists available at ScienceDirect

Engineering Structures

journal homepage: www.elsevier.com/locate/engstruct

# Step-by-step unbalanced force iteration method for cable-strut structure with irregular shape

Qing Ma[c,d], Makoto Ohsaki[d], Zhihua Chen[a,b,c,*], Xiangyu Yan[c,e]

[a] State Key Laboratory of Hydraulic Engineering Simulation and Safety, Tianjin University, Tianjin, China
[b] Key Laboratory of Coast Civil Structure Safety of China Ministry of Education, Tianjin University, Tianjin, China
[c] Department of Civil Engineering, Tianjin University, Tianjin, China
[d] Department of Architecture and Architectural Engineering, Kyoto University, Kyoto, Japan
[e] Architectural Design and Research Institute, Tianjin University, Tianjin, China

ARTICLE INFO

*Keywords:*
Cable-strut structure
Tensegrity
Prestress design
Form finding

ABSTRACT

In the design process of a cable-strut structure, the desired shape is first defined and the prestress can be obtained if the geometry is feasible; otherwise, the geometry must be modified. Thus, the initial step for prestress calculation is to estimate the feasibility of the geometry. In this paper, a method called Unbalanced Force Iteration (UFI) is proposed to remove the unbalanced forces using the equilibrium and stiffness equations. Feasibility of the geometry can be judged by the convergence property of UFI. Self-stress modes can be directly obtained easily through UFI method, if initial geometry is feasible. For structures with infeasible initial geometry, the Step-by-Step UFI, which combines finite element analysis and UFI, is proposed to gradually move the nodes to feasible locations. Three examples of cable domes with feasible geometry and three examples of cable domes with irregular and infeasible initial geometry are presented to verify the ability of UFI and Step-by-Step UFI for designing new irregular and asymmetric cable-strut structures.

## 1. Introduction

Cable-strut structure is a lightweight roof structure covering a large space in the field of civil and architectural engineering. It consists of cables and struts that are stiffened by introducing pretensioning forces in a self-equilibrium state. Therefore, the mechanical properties of a cable-strut structure are similar to those of a tensegrity structure [1,2]. Over the past 60 years, numerous studies have been conducted for analysis and design of tensegrity structures [3,4]. However, a cable-strut structure does not obey the rigorous definition of a tensegrity structure, because a tensegrity structure should be self-standing without support, which is not acceptable for a roof structure.

A long-span cable-strut structure called cable dome was first applied by Geiger in the two Gymnasiums of Seoul Olympics [5]. The structural configuration was then called 'Geiger form'. Levy adopted another form of cable dome consisting of triangle units in the main Gymnasium of Atlanta Olympics (Georgia Dome) [6], which was the largest cable dome up to now. Being a lightweight solution makes it a popular structural system in the design of large span roofs. In the recent three decades more than ten large cable domes have been constructed, including famous works such as the Suncoast Dome, Redbird Arena, Taoyuan County Arena, La Plata Stadium, The Crown Coliseum, etc. [7–12], and many new forms have been proposed and studied [13–18]. However, the majority of existing cable dome projects are in symmetric circular form.

Determination of prestress is the key step in design of cable-strut structures [7], because the stability and stiffness against external loads depend on prestress distribution. Furthermore, *feasible prestress*, or *feasible self-stress mode*, is defined as a set of member forces satisfying the self-equilibrium conditions and the structural requirement such that cables are in tension and struts are in compression [19]. If feasible prestress is able to be obtained, we regard that the geometry is feasible. The process for determination of prestress with given geometry and topology is known as prestress design, while the process of obtaining *feasible geometry* with topological and structural demands is referred to as form-finding [20].

Various methods have been presented for finding feasible prestresses or self-stress modes, e.g., by Zhang et al. [21], Pellegrino [19], Yuan et al. [14], Wang et al. [22], Tran et al. [23], Guo and Jiang [24], Ye et al. [25]. Number of self-stress modes can be easily obtained from the rank of equilibrium matrix [21], which can be computed by Singular Value Decomposition (SVD). If there is only one self-stress mode,

* Corresponding author.
*E-mail address:* zhchen@tju.edu.cn (Z. Chen).

https://doi.org/10.1016/j.engstruct.2018.09.081
Received 10 April 2018; Received in revised form 27 August 2018; Accepted 28 September 2018
0141-0296/ 

feasibility of the geometry can easily checked. If the initial geometry is not feasible, the form-finding process is needed. Many methods have also been proposed for the form-finding problem. Among them, Dynamic Relaxation Method (DRM) [25] and Force Density Method (FDM) [26] are very efficient methods to find the geometry when the unstressed member length or force density is known. For form-finding problems with geometrical and structural constraints, many non-linear or numerical methods have been presented based on FDM/DRM [27–31] or Finite Element Method [23,32–34].

In many projects, the initial surface shape or even geometry have been assigned according to architectural preference and requirement before the structural design starts. Thus, the problems encountered in the practical design process of asymmetric or irregular cable domes are (1) a more efficient and direct way for estimation of feasibility of the given initial geometry is needed; (2) when the initial geometry is infeasible, a simple form-finding method is needed for structural designers to find the feasible geometry closest to the initial one.

In this paper, a method called *Unbalanced Force Iteration (UFI)* is first proposed. The geometrical feasibility can be estimated by the convergence property of UFI. This method is conducted by a simple iteration with equilibrium matrix and linear stiffness matrix only, which are available in any structural analysis software. Self-stress modes and the static indeterminacy, which is equal to the number of independent self-stress modes, can be obtained in a simple manner, if geometry is feasible. This process avoids conducting singular value decomposition of equilibrium matrix to obtain its rank, which is difficult for a structural engineer. When initial geometry is infeasible, the *Step-by-Step UFI*, which combines UFI and Finite Element Analysis (FEA), is proposed to gradually move the nodes to feasible locations. Feasible geometry that is close to keeps the initial surface shape and close to the given coordinates can be found. The efficiency of the proposed methods is demonstrated in numerical examples of cable-strut structures with various symmetric and asymmetric shapes. Evaluation of feasibility, obtainment of self-stress modes and form finding can be integrated together. The whole method is also more intuitive and easier for engineers to master and apply in designing process of real projects.

## 2. Basic formulations

Basic formulations of equilibrium equations and stiffness matrices are presented for prestress design and form-finding analysis of cable-strut structures.

### 2.1. Equilibrium equation

Let $m$ and $n$ denote the numbers of elements and nodes, respectively. Connectivity of the structure can be defined by the connectivity matrix $\boldsymbol{C} \in R^{n\times m}$. If the $k$th element starts at node $i$ and ends at node $j$, the $(k, e)$ component $C_{k,e}$ of matrix $\boldsymbol{C}$ is defined as:

$$C_{k,e} = \begin{cases} +1 & (e = i) \\ -1 & (e = j) \\ 0 & (\text{others}) \end{cases} \tag{1}$$

Let $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ $(\in R^{n\times 1})$ denote the coordinate vectors of nodes in $x$-, $y$- and $z$-directions, respectively. The vectors of coordinate differences in $x$-, $y$- and $z$-directions of the two end nodes of members are denoted by $\boldsymbol{u} = (u_1, ..., u_m)^{\mathrm{T}}$, $\boldsymbol{v} = (v_1, ..., v_m)^{\mathrm{T}}$ and $\boldsymbol{w} = (w_1, ..., w_m)^{\mathrm{T}}$, respectively, which are calculated by

$$\begin{cases} \boldsymbol{u} = \boldsymbol{Cx} \\ \boldsymbol{v} = \boldsymbol{Cy} \\ \boldsymbol{w} = \boldsymbol{Cz} \end{cases} \tag{2}$$

The coordinate difference matrices $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{W}$ $(\in R^{m\times m})$ are defined as

$$\begin{cases} \boldsymbol{U} = \mathrm{diag}(\boldsymbol{u}) \\ \boldsymbol{V} = \mathrm{diag}(\boldsymbol{v}) \\ \boldsymbol{W} = \mathrm{diag}(\boldsymbol{w}) \end{cases} \tag{3}$$

Let $\boldsymbol{l} = (l_1, ..., l_m)^{\mathrm{T}}$ denote the vector consisting of length of each element including cables and struts. The $k$th component $l_k$ is computed as

$$l_k = \sqrt{u_k^2 + v_k^2 + w_k^2} \tag{4}$$

The element length matrix $\boldsymbol{L}$ is defined as

$$\boldsymbol{L} = \mathrm{diag}(\boldsymbol{l}) \tag{5}$$

Then the equilibrium matrix $\boldsymbol{D}^0 \in R^{3n\times m}$ including the support degree of freedom is obtained as

$$\boldsymbol{D}^0 = \begin{bmatrix} \boldsymbol{C}^{\mathrm{T}}\boldsymbol{U}\boldsymbol{L}^{-1} \\ \boldsymbol{C}^{\mathrm{T}}\boldsymbol{V}\boldsymbol{L}^{-1} \\ \boldsymbol{C}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{L}^{-1} \end{bmatrix} \tag{6}$$

Let $f$ denote the number of degrees of freedom of the structure. Equilibrium matrix $\boldsymbol{D}^0$ is reduced to $\boldsymbol{D} \in R^{f\times m}$ after removing the rows corresponding to the fixed displacement components. Self-equilibrium conditions of the cable-strut structure is formulated with respect to the vector $\boldsymbol{t} \in R^{m\times 1}$ consisting of prestress of each element as

$$\boldsymbol{Dt} = \boldsymbol{0} \tag{7}$$

### 2.2. Linear stiffness matrix

The linear elastic properties of pretensioned cable-strut structures are the same as those of trusses, when cables are appropriately tensioned. Therefore, we derive the linear stiffness matrix of trusses below. Let $\boldsymbol{k}_e \in R^{m\times 1}$ denote the element stiffness vector. The $k$th component $k_e^k$ can be calculated by

$$k_e^k = \frac{EA_k}{l_k} \tag{8}$$

where $E$ is Young's modulus and $A_k$ is the cross-sectional area of the $k$th element. A diagonal matrix $\boldsymbol{K}_e \in R^{m\times m}$ of element stiffness is defined as

$$\boldsymbol{K}_e = \mathrm{diag}(\boldsymbol{k}_e) \tag{9}$$

Let $\boldsymbol{d} \in R^{f\times 1}$ denote the nodal displacement vector of the structure subjected to the external load vector $\boldsymbol{F} \in R^{f\times 1}$. Elongation vector $\Delta\boldsymbol{l} \in R^{m\times 1}$ of members is calculated by

$$\Delta\boldsymbol{l} = \boldsymbol{Bd} \tag{10}$$

where $\boldsymbol{B} \in R^{m\times 3n}$ is the compatibility matrix, which is the transpose of the equilibrium matrix $\boldsymbol{D}$ as

$$\boldsymbol{B} = \boldsymbol{D}^{\mathrm{T}} \tag{11}$$

Then, the global linear stiffness matrix $\boldsymbol{K}_E \in R^{f\times f}$ after removing the support degrees of freedom is obtained as

$$\boldsymbol{K}_E = \boldsymbol{D}\boldsymbol{K}_e\boldsymbol{D}^{\mathrm{T}} \tag{12}$$

The nodal displacement vector $\boldsymbol{d}$ under nodal loads $\boldsymbol{F}$ can be obtained from the following equation:

$$\boldsymbol{K}_E\boldsymbol{d} = \boldsymbol{F} \tag{13}$$

The resulting internal force can be calculated by

$$\boldsymbol{t} = \Delta\boldsymbol{l}\circ\boldsymbol{k}_e = \boldsymbol{K}_e\Delta\boldsymbol{l} = \boldsymbol{K}_e\boldsymbol{D}^{\mathrm{T}}\boldsymbol{d} \tag{14}$$

with ( ∘ ) denoting Hadamard product [35], or element-wise product, of two vectors or matrices.