Journal of Combinatorial Theory, Series B ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Planar graphs are 9/2-colorable

Daniel W. Cranston [a], Landon Rabern [b]

[a] Virginia Commonwealth University, Richmond, VA, United States
[b] LBD Data Solutions, Lancaster, PA, United States

A R T I C L E   I N F O

Article history:
Received 19 January 2015
Available online xxxx

Keywords:
4 Color Theorem
Coloring
Planar
Homomorphism
Fractional coloring

A B S T R A C T

We show that every planar graph G has a 2-fold 9-coloring. In particular, this implies that G has fractional chromatic number at most $\frac{9}{2}$. This is the first proof (independent of the 4 Color Theorem) that there exists a constant $k < 5$ such that every planar G has fractional chromatic number at most k.

1. Introduction

All graphs in this paper are finite, loopless, and simple (parallel edges are forbidden). To fractionally color a graph G, we assign to each independent set in G a nonnegative weight, such that for each vertex v the sum of the weights on the independent sets containing v is 1. A graph G is *fractionally* k*-colorable* if G has such an assignment of weights where the sum of the weights is at most k. The minimum k such that G is fractionally k-colorable is its *fractional chromatic number*, denoted $\chi_f(G)$. (If we restrict the weight on each independent set to be either 0 or 1, then we return to the standard definition of chromatic number.) In 1997, Scheinerman and Ullman [13, p. 75] succinctly described the state of the art for fractionally coloring planar graphs. Not much has changed since then.

E-mail addresses: dcranston@vcu.edu (D.W. Cranston), landon.rabern@gmail.com (L. Rabern).

https://doi.org/10.1016/j.jctb.2018.04.002
0095-8956/

The fractional analogue of the four-color theorem is the assertion that the maximum value of $\chi_f(G)$ over all planar graphs G is 4. That this maximum is no more than 4 follows from the four-color theorem itself, while the example of K_4 shows that it is no less than 4. Given that the proof of the four-color theorem is so difficult, one might ask whether it is possible to prove an interesting upper bound for this maximum without appeal to the four-color theorem. Certainly $\chi_f(G) \leq 5$ for any planar G, because $\chi(G) \leq 5$, a result whose proof is elementary. But what about a simple proof of, say, $\chi_f(G) \leq \frac{9}{2}$ for all planar G? The only result in this direction is in a 1973 paper of Hilton, Rado, and Scott [7] that predates the proof of the four-color theorem; they prove $\chi_f(G) < 5$ for any planar graph G, although they are not able to find any constant $c < 5$ with $\chi_f(G) < c$ for all planar graphs G. This may be the first appearance in print of the invariant χ_f.

In Section 2, we give exactly what Scheinerman and Ullman asked for—a simple proof that $\chi_f(G) \leq \frac{9}{2}$ for every planar graph G. In fact, this result is an immediate corollary of a stronger statement in our main theorem. Before we can express it precisely, we need another definition. A *k-fold ℓ-coloring* of a graph G assigns to each vertex a set of k colors, such that adjacent vertices receive disjoint sets, and the union of all sets has size at most ℓ. If G has a k-fold ℓ-coloring, then $\chi_f(G) \leq \frac{\ell}{k}$. To see this, consider the ℓ independent sets induced by the color classes; assign to each of these sets the weight $\frac{1}{k}$. Now we can state the theorem.

Main Theorem. *Every planar graph G has a 2-fold 9-coloring. In particular, $\chi_f(G) \leq \frac{9}{2}$.*

In an intuitive sense, the Main Theorem sits somewhere between the 4 Color Theorem and the 5 Color Theorem. It is certainly implied by the former, but it does not immediately imply the latter. The *Kneser graph* $K_{n:k}$ has as its vertices the k-element subsets of $\{1, \dots, n\}$ and two vertices are adjacent if their corresponding sets are disjoint. Saying that a graph G has a 2-fold 9-coloring is equivalent to saying that it has a homomorphism to the Kneser graph $K_{9:2}$. To claim that a coloring result for planar graphs is between the 4 and 5 Color Theorems, we would like to show that every planar graph G has a homomorphism to a graph H, such that H has clique number 4 and chromatic number 5. (Since K_4 can map into H, we know that H has clique number at least 4. And clique number less than 5 means our result is something more than just the 5 Color Theorem. The fact that H has chromatic number 5 means that our result implies the 5 Color Theorem.) Unfortunately, $K_{9:2}$ is not such a graph. It is easy to see that $\omega(K_{n:k}) = \lfloor n/k \rfloor$; so $\omega(K_{9:2}) = 4$, as desired. However, Lovász [9] showed that $\chi(K_{n:k}) = n - 2k + 2$; thus $\chi(K_{9:2}) = 9 - 2(2) + 2 = 7$. Fortunately, we can easily overcome this problem.

The *categorical product* (or *universal product*) of graphs G_1 and G_2, denoted $G_1 \times G_2$ is defined as follows. Let $V(G_1 \times G_2) = \{(u, v) | u \in V(G_1) \text{ and } v \in V(G_2)\}$; now (u_1, v_1) is adjacent to (u_2, v_2) if $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$. Let $H = K_5 \times K_{9:2}$. It is well-known [6] that if a graph G has a homomorphism to each of graphs G_1 and G_2,

Please cite this article in press as: D.W. Cranston, L. Rabern, Planar graphs are 9/2-colorable, J. Combin. Theory Ser. B (2018), https://doi.org/10.1016/j.jctb.2018.04.002