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The smallest eigenvalues of Hamming graphs, Johnson graphs and other distance-regular graphs with classical parameters

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ABSTRACT

We prove a conjecture by Van Dam & Sotirov on the smallest eigenvalue of (distance-j) Hamming graphs and a conjecture by Karloff on the smallest eigenvalue of (distance-j) Johnson graphs. More generally, we study the smallest eigenvalue and the second largest eigenvalue in absolute value of the graphs of the relations of classical P- and Q-polynomial association schemes.

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1. Introduction

In this paper we study the smallest eigenvalue as well as the second largest one in absolute value of the adjacency matrix of several important families of graphs, all belonging to the classical P- and Q-polynomial association schemes [2, Chapter 6].

The most well-known example of a P-polynomial association scheme is the Hamming scheme. We investigate the eigenvalues of the graphs that have the vectors in $\mathbb{F}_q^d$ as vertices and two vertices are adjacent if they have Hamming distance j. The smallest eigenvalues are important for determining the max-cut of certain graphs in the Hamming scheme. These graphs provide examples where the performance ratio of the Goemans–Williamson algorithm is tight [1]. The smallest eigenvalues are also used for determining the max-k-cut [6] and the chromatic number of the graphs in the Hamming scheme [6].

The second important scheme belonging to the family of P-polynomial association schemes is the Johnson scheme. Here the vertices are the d-subsets of $\{1, 2, \ldots, n\}$. We investigate the eigenvalues of the graph where two d-sets are adjacent if they differ in exactly j elements. As for the Hamming scheme, these graphs provide examples for which the performance ratio of the Goemans–Williamson algorithm is tight and their smallest eigenvalues are central for determining their max-cuts [20]. These graphs are also important for investigating subsets with exactly one forbidden intersection, a variation of the classical Erdős–Ko–Rado theorem due to Frankl and Füredi [18].

The other graphs under investigation are Grassmann graphs, dual polar graphs, and various forms graphs, most prominently the bilinear forms graphs. Again, the smallest eigenvalues can be used to investigate the max-cuts and intersecting families in these graphs. The P-polynomial graphs obtain their importance from various applications. For example, Grassmann graphs are of interest due to their applications in network coding theory [26] and their role in the recent proof of the 2-to-2-games conjecture [21].

In the following we give a short summary of our main results on the specific families.

1.1. Hamming graphs

Let $q \geq 2, d \geq 1$ be integers. Let Q be a set of size q. The Hamming scheme $H(d, q)$ is the association scheme with vertex set Q^d, and as relation the Hamming distance. The $d+1$ relation graphs $H(d, q, j)$, where $0 \leq j \leq d$, have vertex set Q^d, and two vectors of length d are adjacent when they differ in j places.

The eigenmatrix P of $H(d, q)$ has entries $P_{ij} = K_j(i)$, where

$$K_j(i) = \sum_{h=0}^{j} (-1)^h (q-1)^{j-h} \binom{i}{h} \binom{d-i}{j-h}.$$

The eigenvalues of the graph $H(d, q, j)$ are the numbers in column j of P, so are the numbers $K_j(i)$, $0 \leq i \leq d$. The graph $H(d, q, j)$ is regular of degree $K_j(0) = (q-1)^j \binom{d}{j}$,