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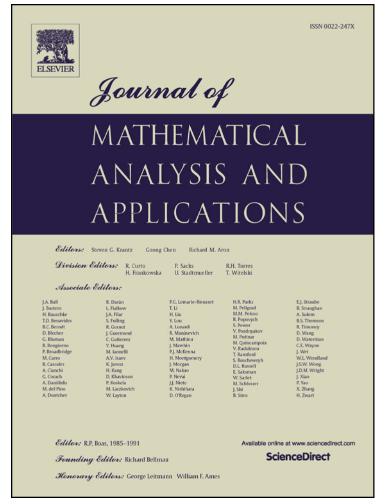

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Monotonicity Properties and Inequalities for the Generalized Elliptic Integral of the First Kind *

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Abstract: The authors obtain some monotonicity and concavity properties, and asymptotically sharp bounds for the generalized elliptic integral $\mathscr{K}_a(r)$ of the first kind and its special case $\mathscr{K}(r)$, the complete elliptic integral of the first kind, by studying the monotonicity and concavity properties of certain combinations defined in terms of the Gaussian hypergeometric function $F(a, 1-a; 1; x)$ and elementary functions, thus improving some well-known results.

Key Words: generalized elliptic integrals of the first kind; complete elliptic integrals of the first kind; Gaussian hypergeometric function; Ramanujan constant (or $R-$function); monotonicity; concavity; sharp inequalities

Mathematics Subject Classification: 33C05, 33E05, 26D20.

1 Introduction

Throughout this paper, for the real parameters a, b, c with $c \neq 0, -1, -2, \cdots$, we let

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)n!} x^n, \ x \in (-1, 1), \tag{1.1}$$

denote the Gaussian hypergeometric function as usual, where (a, n) is the shifted factorial function $(a, n) \equiv a(a+1)\cdots(a+n-1)$ for $n \in \mathbb{N}$, and $(a, 0) = 1$ for $a \neq 0$. $F(a, b; c; x)$ is said to be zero-balanced if $c = a + b$. It is well known that $F(a, b; c; x)$ is widely applied in many fields of mathematics and some other disciplines, and many other special functions in mathematical physics and even some elementary functions are particular or limiting cases of this function. (Cf. [2, 4, 9, 12, 14–16, 18, 24, 33, 35, 48].)

For $a \in (0, 1)$, the generalized elliptic integrals $\mathscr{K}_a(r)$ and $\mathscr{K}_a'(r)$ of the first kind on $[0, 1]$ are defined as

$$\begin{cases} \mathscr{K}_a(r) = \frac{\pi}{2} F\left(a, 1-a; 1; r^2\right), \ r \in (0, 1), \\ \mathscr{K}_a'(r) = \mathscr{K}_a(r'), \\ \mathscr{K}_a(0) = \frac{\pi}{2}, \ \mathscr{K}_a(1^-) = \infty. \end{cases} \tag{1.2}$$

Here and hereafter we always let $r' = \sqrt{1-r^2}$ for $r \in [0, 1]$. If $a = 1/2$, then $\mathscr{K}_a(r)$ and $\mathscr{K}_a'(r)$ reduce to the following well-known complete elliptic integrals $\mathscr{K}(r)$ and $\mathscr{K}'(r)$ of the first kind

$$\begin{cases} \mathscr{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \left(1 - r^2 \sin^2 t\right)^{-1/2} dt, \ r \in (0, 1), \\ \mathscr{K}'(r) = \mathscr{K}(r'), \\ \mathscr{K}(0) = \frac{\pi}{2}, \ \mathscr{K}(1^-) = \infty. \end{cases} \tag{1.3}$$

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