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# An algebra of Stein operators

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ABSTRACT

We build upon recent advances on the distributional aspect of Stein's method to propose a novel and flexible technique for computing Stein operators for random variables that can be written as products of independent random variables. We show that our results are valid for a wide class of distributions including normal, beta, variance-gamma, generalized gamma and many more. Our operators are $k$th degree differential operators with polynomial coefficients; they are straightforward to obtain even when the target density bears no explicit handle. As an application, we derive a new formula for the density of the product of $k$ independent symmetric variance-gamma distributed random variables.

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## 1. Introduction

In 1972, Charles Stein (1920–2016) [41] introduced a powerful method for estimating the error in normal approximations. The method was adapted to the Poisson distribution by Louis Chen in [8], and has since been extended to a very broad family of probability distributions. The general procedure for a given target distribution $p$ is as follows. In the first step, one obtains a suitable operator $A$ acting on a class of test functions $\mathcal{F}$ such that $\mathbb{E}[Af(X)] = 0$ for all $f \in \mathcal{F}$; the operator $A$ is called a *Stein operator* for $p$. For continuous distributions, $A$ is a typically a differential operator; for the standard normal distribution, the classical operator is $Af(x) = f'(x) - xf(x)$. One then considers the so-called *Stein equation*

$$Af_h(x) = h(x) - \mathbb{E}h(X), \tag{1}$$

where $h$ is a real-valued test function. If $A$ is well chosen then, for a given $h$, the Stein equation (1) can be solved for $f_h$. The second step of the method consists of obtaining this solution and then bounding

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appropriate lower order derivatives. Evaluating both sides of (1) at a random variable of interest $W$ and taking the supremum over all $h$ in some class of functions $\mathcal{H}$ leads to the estimate

$$d_{\mathcal{H}}(\mathcal{L}(W),\mathcal{L}(X)) := \sup_{h\in\mathcal{H}} |\mathbb{E}h(W) - \mathbb{E}h(X)| \leq \sup_{f_h} |\mathbb{E}[Af_h(W)]|, \tag{2}$$

where the final supremum is taken over all $f_h$ that solve (1). The third and final step of the method involves developing appropriate strategies for bounding the expectation on the right hand side of (2). This is of interest because many important probability metrics (such as the Kolmogorov and Wasserstein metrics) are of the form $d_{\mathcal{H}}(\mathcal{L}(W),\mathcal{L}(X))$. Moreover, in many settings bounding the expectation $\mathbb{E}[Af_h(W)]$ is relatively tractable, and as a result Stein's method has found application in disciplines as diverse as random graph theory [5], number theory [22], statistical mechanics [13] and quantum mechanics [29]. We refer to the survey paper [37] as well as to the monographs [30,9] for a deeper look into some of the fruits of Charles Stein's seminal insights, particularly in the case where the target is the normal distribution.

The linchpin of the method is the operator $A$ whose properties are crucial to the success of the whole enterprise. In the sequel, we concentrate exclusively on *differential* Stein operators (some operators in the literature are integral or even fractional, see e.g. [43,3]) and adopt the following lax definition:

**Definition 1.1.** A linear differential operator $A$ acting on a class $\mathcal{F}$ of functions is a *Stein operator* for $X$ if (i) $Af \in L^1(X)$ and (ii) $\mathbb{E}\left[Af(X)\right] = 0$ for all $f \in \mathcal{F}$.

There are infinitely many Stein operators for any given target distribution. For instance, if the distribution is known (even if only up to a normalizing constant) then the "canonical" theory from [26] applies, leading to entire families of operators. This approach provides natural first order polynomial operators e.g. for target distributions which belong to the Pearson family [38] or which satisfy a diffusive assumption [11,25]. In some cases, one may rather apply a duality argument. For instance the p.d.f. $\gamma(x) = (2\pi)^{-1/2}\mathrm{e}^{-x^2/2}$ of the standard normal distribution satisfies the first order ODE $\gamma'(x)+x\gamma(x) = 0$ leading, by integration by parts, to the already mentioned operator $Af(x) = f'(x) - xf(x)$. This is particularly useful for densities defined implicitly via ODEs. Such are by no means the only methods for deriving differential Stein operators and, for any given $X$, one can easily determine an entire ecosystem of Stein operators, leading to the natural question of *which operator to choose*. One natural way to sieve through the available options is to further impose that the chosen operator be characterizing for $X$, i.e. that if some $Y$ enjoys the property that $\mathbb{E}[f(Y)] = 0$ for all $f \in \mathcal{F}$, then $Y{=}X$ (equality in law). Such requirements often do not suffice and will not be imposed here; our focus will rather be on another crucial quality of a "good" Stein operator: tractability. More precisely, we will focus solely on Stein operators which satisfy the next definition.

**Definition 1.2.** We call a Stein operator *polynomial* if it can be written as a finite sum $A = \sum_{i,j} a_{ij}M^iD^j$ for real coefficients $a_{ij} \in \mathbb{R}$, with $M(f) = (x \mapsto xf(x))$ and $D(f) = (x \mapsto f'(x))$.

Except in the most basic cases, determining polynomial Stein operators is not an easy task. Interestingly, many densities do not admit a first order polynomial Stein operator and it is necessary to consider higher order operators: [15] obtains a second order operator for the entire family of variance-gamma distributions (see also [14] and [17]), [36] obtain a second order Stein operator for the Laplace distribution, and [34] obtain a second order operator for the PRR distribution, which has a density that can be expressed in terms of the Kummer $U$ function. If the p.d.f. of $X$ is defined in terms of special functions (Kummer $U$, Meijer $G$, Bessel, etc.) which are themselves defined as solutions to explicit $d$th order differential equations then the duality approach shall yield a tractable differential operator with explicit coefficients.

In many cases, the target distribution is not even defined analytically in terms of its distribution but rather probabilistically, as a statistic (sum, product, quotient) of independent contributions. Explicit knowledge of the density of such random variables is then generally unavailable and, in order to obtain polynomial

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