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Dynamic Picone's identity and its applications ☆

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ABSTRACT

We prove some Picone-type identities and inequalities for a class of first-order nonlinear dynamic systems and derive various weighted inequalities of Wirtinger type and Hardy type on time scales. As applications we study oscillatory and related properties of these systems including Reid's roundabout theorem on disconjugacy, Sturm's separation and comparison theorems, as well as a variational method in the oscillation theory.

1. Introduction

In 1910, Picone [27] discovered the following identity

$$\left(puu' - Pu^2\frac{v'}{v}\right)' = (p-P)(u')^2 + (Q-q)u^2 + P\left[v\left(\frac{u}{v}\right)'\right]^2 + \frac{u}{v}(v\ell_2[u] - u\mathcal{L}_2[v]), \tag{1.1}$$

which was used to obtain Sturm's comparison theorem for ordinary differential operators $\ell_2[u] = (pu')' + qu$ and $\mathcal{L}_2[v] = (Pv')' + Qv$. Picone's identity (1.1) turns out to be a useful tool in studying oscillation theory of linear differential equations as well as in establishing integral inequalities involving functions and their derivatives such as Wirtinger's and Hardy's inequalities.

Since then, there has been an increasing interest in developing Picone-type identities and inequalities. Indeed, Jaroš and Kusano [20] extended Picone's identity (1.1) to the case of second-order half-linear differential operators $\ell_\alpha[u] = (p\varphi_\alpha(u'))' + q\varphi_\alpha(u)$ and $\mathcal{L}_\alpha[v] = (P\varphi_\alpha(v'))' + Q\varphi_\alpha(v)$, where $\alpha > 0$, $\varphi_\alpha(t) = |t|^\alpha \operatorname{sgn}(t)$, and applied it to study Sturm's theory for both homogeneous $\ell_\alpha[u] = 0$ and nonhomogeneous

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$\ell_\alpha[u] = f$ equations. The discrete and dynamic versions of Jaroš and Kusano's result were obtained later by Řehák [28] and by Agarwal, Bohner and Řehák [3], respectively.

Recently, Jaroš [18] established a weaker Picone-type identity for the operator $\mathcal{L}_\alpha[v]$ as follows

$$\left(\frac{|x|^{\beta+1}P\varphi_\alpha(v')}{\varphi_\beta(v)}\right)' = P|v'|^{\alpha-\beta}\left[|x'|^{\beta+1} - \Phi_\beta\left(x', \frac{xv'}{v}\right)\right] + \frac{|x|^{\beta+1}}{\varphi_\beta(v)}[\mathcal{L}_\alpha[v] - Q\varphi_\alpha(v)], \tag{1.2}$$

where $\beta > 0$, x is an absolutely continuous function and

$$\Phi_\beta(t,s) = |t|^{\beta+1} + \beta|s|^{\beta+1} - (\beta+1)t\varphi_\beta(s), \quad t,s \in \mathbb{R}. \tag{1.3}$$

He then used (1.2) and the positive semi-definiteness of Φ_β (as a consequence of Young's inequality) to prove that if v is a positive solution of the second-order differential inequality

$$\mathcal{L}_\alpha[v] + \lambda R\varphi_\beta(v) \le 0 \quad (\lambda \in \mathbb{R}), \tag{1.4}$$

then

$$\int_a^b (Q|v|^{\alpha-\beta} + \lambda R)|x|^{\beta+1}dt \le \int_a^b P|v'|^{\alpha-\beta}|x'|^{\beta+1}dt + S_a(x;v) - S_b(x;v), \tag{1.5}$$

where

$$S_a(x;v) = \lim_{t\to a^+}\left(\frac{|x|^{\beta+1}P\varphi_\alpha(v')}{\varphi_\beta(v)}\right)(t), \quad S_b(x;v) = \lim_{t\to b^-}\left(\frac{|x|^{\beta+1}P\varphi_\alpha(v')}{\varphi_\beta(v)}\right)(t).$$

An inequality of Wirtinger type (1.5) generalizes inequalities obtained by Diaz and Metcalf [14], Lee et al. [24] as well as by Swanson [33].

Most recently, Jaroš [19] and Tiryaki [34] gave some Picone-type identities and inequalities for the following nonlinear differential system

$$\begin{cases} u' = Au + B\varphi_{1/\alpha}(v), \\ v' = -C\varphi_\alpha(u) - Dv, \end{cases} \tag{1.6}$$

and used them to derive Wirtinger-type inequalities analogous to (1.5). However, when applying these inequalities to study the existence and distribution of zeros of the first component of the solution of (1.6) (see [19, Corollary 3.3] and [34, Corollary 2.6, Theorems 2.8 and 2.10]), they did not verify the conditions of the inequalities. Lemma 3.2 in this paper will fill in this gap.

What we propose in this paper is a unification and an extension of the above works and their discrete analogues to time scales. One may ask whether this is still true if we consider such problems for equations and systems on an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. Recall here the first effort was due to Agarwal et al. [6] who proved some dynamic Wirtinger's inequalities by considering the Δ-differential inequality

$$(Pu^\Delta)^\Delta + Qu^\sigma + \lambda Ru^\sigma \le 0 \quad \text{on} \quad \mathbb{I} \cap [a, \bar{\rho}(b)), \tag{1.7}$$

where λ is a real parameter, P, Q and R are real-valued rd-continuous functions on $\mathbb{I}$ with $P > 0$, and