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The roll call interpretation of the Shapley value

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HIGHLIGHTS

- Roll call interpretation of the Shapley value permits diverse cooperation patterns.
- A pivotal vote in a roll call seals a proposal's fate in either way.
- The probability of being pivotal is commonly viewed as a player's voting power.
- The Shapley value equals pivot probabilities if and only if votes are exchangeable.

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ABSTRACT

The Shapley value is commonly illustrated by roll call votes in which players support or reject a proposal in sequence. If all sequences are equiprobable, a voter's Shapley value can be interpreted as the probability of being pivotal, i.e., to bring about the required majority or to make this impossible for others. We characterize the joint probability distributions over cooperation patterns that permit this roll call interpretation: individual votes may be interdependent but must be exchangeable.

1. Introduction

A player's *Shapley value* equals its expected contribution to surplus creation if full cooperation among players is established in random order. Going back to Shapley (1953) and Shapley and Shubik (1954), this is often illustrated by voting games: shareholders, delegates to a council, parties, etc. cast their respective voting weight in favor of a proposal one after another. If player i's vote is the first to reach the required majority threshold, it 'swings' the status of the coalition S of earlier supporters from losing ($v(S) = 0$) to winning ($v(S \cup \{i\}) = 1$); i is then attributed a 'marginal contribution' of $v(S\cup\{i\})-v(S) = 1$. Averaging these contributions across all equiprobable voting sequences yields i's Shapley value $\varphi_i(v)$. It is equal to the probability that i is decisive for passing a proposal. This is commonly interpreted as voting power and also called i's *Shapley–Shubik index (SSI)*.

The implicit assumption in this well-known roll call interpretation of Shapley value and SSI is that all voters support the proposal, i.e., every player joins the coalition either sooner or later. This was criticized early on, e.g., by Luce and Raiffa (1957, p. 255). It is still not widely known that the roll call interpretation of the Shapley value extends considerably beyond uniform "yes" votes.

Namely, a voter can also be decisive for rejecting a proposal by voting "no" and being first to ensure that the required majority cannot be met. In general, we say player i is *pivotal* in a given voting sequence if the collective decision may still go either way before i's vote but becomes fully determined by it. Already Mann and Shapley (1960, p. 4; 1964, p. 153) observed that player i's SSI equals i's pivot probability if all players vote in a mutually independent way with a common probability $x \in [0, 1]$ for "yes", not just when $x = 1$ or 0. This was first explicitly proven in Felsenthal and Machover (1996).

But $\varphi_i(v)$'s roll call interpretation applies even more generally: it is sufficient that players' votes are *exchangeable*, so possibly dependent. This can be deduced from combinatorial results by Hu (2006, Prop. 4). We give a short non-combinatorial proof here. Our main objective, however, is to show that exchangeability is necessary, too: i's Shapley value equals its pivot probability in roll

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call votes with random order *if and only if* players' cooperation decisions are exchangeable.

A characterization of when pivotality in role calls reduces to the Shapley value is of interest beyond committee decisions: binary threshold structures similar to voting appear in diverse contexts. Think, e.g., of dichotomous stability assessments in which loans that are either performing or non-performing play the role of votes and exceeding a given quota of non-performing loans reflects insolvency. And if the usual definition of i's marginal contribution is extended to reflect also the reduction of creatable surplus if i refuses to cooperate, then the roll call interpretation of the Shapley value extends to general coalitional games without full cooperation too.

2. Preliminaries

Consider a set $N = \{1, \dots, n\}$ of $n > 0$ players. A *coalitional game* $v: 2^N \to \mathbb{R}$ with $v(\varnothing) = 0$ maps each *coalition* $S \subseteq N$ of cooperating players to a real number, typically interpreted as a surplus that increases from zero to $v(N)$ as more players cooperate. In voting applications, $i \in S$ reflects a "yes" vote by player i. Then the focus is on *simple (voting) games* with $v(S) \in \{0, 1\}$: $v(S) = 1$ identifies passage of a proposal, $v(\varnothing) = 0$, $v(N) = 1$, and $S \subseteq T \Rightarrow v(S) \leq v(T)$. Simple games u_T defined by $u_T(S) = 1 \Leftrightarrow T \subseteq S$ for given $\varnothing \neq T \subseteq N$ are called *unanimity games* and form a basis of the vector space of coalitional games.

Values are operators that map coalitional games to $\mathbb{R}^n$ and thereby suggest an allocation of $v(N)$, indicate the distribution of voting power, etc. A value ψ is called *linear* if $\psi(\alpha \cdot u + \beta \cdot v) = \alpha\cdot\psi(u)+\beta\cdot\psi(v)$ for all constants $\alpha, \beta \in \mathbb{R}$ and all coalitional games u, v on the same set N of agents, where $(\alpha \cdot u + \beta \cdot v)(S) = \alpha \cdot u(S)+\beta\cdot v(S)$ for all $S \subseteq N$. ψ is called *efficient* if $\sum_{i\in N} \psi_i(v) = v(N)$. A player $i \in N$ satisfying $v(S) = v(S\cup\{i\})$ for all $S \subseteq N\setminus\{i\}$ is called *null*. If $\psi_i(v) = 0$ whenever i is a null player in v, then ψ satisfies the *null player property*. Players $i, j \in N$ with $v(S\cup\{i\}) = v(S\cup\{j\})$ for all $S \subseteq N \setminus \{i, j\}$ are called *equivalent*. ψ is *symmetric* if $\psi_i(v) = \psi_j(v)$ whenever $i, j \in N$ are equivalent in v.

Denote the set of all permutations of N by $\mathcal{S}_n$ and let P_i^π be the set of all agents that precede i in order $\pi \in \mathcal{S}_n$. Then the *Shapley value* φ is defined by

$$\varphi_i(v) = \frac{1}{n!} \cdot \sum_{\pi\in\mathcal{S}_n} \left[v\left(P_i^\pi \cup \{i\}\right) - v\left(P_i^\pi\right)\right] \quad \text{for all } i \in N. \tag{1}$$

This can also be written and more efficiently be computed as

$$\varphi_i(v) = \sum_{S\subseteq N\setminus\{i\}} \frac{|S|! \cdot (n - |S| - 1)!}{n!} \cdot [v(S \cup \{i\}) - v(S)], \tag{2}$$

i.e., by summing only over 2^{n-1} coalitions instead of $n!$ permutations. Shapley (1953) proved that φ is the unique value that satisfies efficiency, linearity, symmetry, and the null player property.

Shapley also gave Eq. (1) a roll call interpretation: assume that all players consent to cooperate one after the other. Given an ordering $\pi \in \mathcal{S}_n$, player i's effect on the joint surplus at the time when i decides is $v\left(P_i^\pi \cup \{i\}\right) - v\left(P_i^\pi\right)$. Considering all orderings to be equiprobable and taking expectations gives Eq. (1).

Shapley and Shubik (1954, p. 789) mentioned for simple games that one can equivalently arrive at $\varphi_i(v)$ assuming all players vote "no". If a player decides not to cooperate in a coalitional game, then formation of the grand coalition N is blocked; the player rescinds some surplus that might potentially be created. At the time of choosing not to cooperate, the size of this destructive effect of player i's non-cooperation is

$$v\left(N \setminus P_i^\pi\right) - v\left(N \setminus \left(P_i^\pi \cup \{i\}\right)\right) = v^*\left(P_i^\pi \cup \{i\}\right) - v^*\left(P_i^\pi\right), \tag{3}$$

where $v^*(S) := v(N) - v(N \setminus S)$ for all $S \subseteq N$ defines the *dual game* of v and $\varphi(v^*) = \varphi(v)$.

Allowing cooperation ("yes") by some players and non-cooperation ("no") by others gives rise to a *generalized roll call model* that was introduced by Mann and Shapley (1960, p. 4; 1964, p. 153) and taken up by Felsenthal and Machover (1996): an ordering π of players is determined; each player $i \in N$ is called in order; when called, i decides either to cooperate or not. Denoting the resulting final sets of cooperators or supporters of a motion by S and the non-cooperators by $\bar{S} := N \setminus S$, the actual surplus created is $v(S)$; the potential surplus rescinded is $v^*(\bar{S}) = v(N) - v(S)$. A particular instance of a roll call will be referred to as $\mathcal{R} = (\pi, S)$ for $\pi \in \mathcal{S}_n$ and $S \in 2^N$.

To assess the effect of a given player i in this process of (non-)creation in game v, let $\mathcal{Y}(\mathcal{R}, i)$ denote the set of cooperative players $j \in S$ that precede player i. Similarly, let $\mathcal{N}(\mathcal{R}, i)$ collect all uncooperative players $j \in \bar{S}$ that precede i. We can then define the *marginal contribution* of player i in roll call $\mathcal{R}$ for game v as

$$M(v, \mathcal{R}, i) = \begin{cases} v(\mathcal{Y}(\mathcal{R}, i) \cup \{i\}) - v(\mathcal{Y}(\mathcal{R}, i)) & \text{if } i \in S, \\ v^*(\mathcal{N}(\mathcal{R}, i) \cup \{i\}) - v^*(\mathcal{N}(\mathcal{R}, i)) & \text{if } i \in \bar{S}. \end{cases} \tag{4}$$

For a simple game v, $M(v, \mathcal{R}, i) \in \{0, 1\}$ and $M(v, \mathcal{R}, i) = 1$ if and only if player i is pivotal in $\mathcal{R}$: fate of a given proposal is still open before i's vote but sealed by i's decision.

Player i's overall effect or power in game v can be captured by computing its expected marginal contribution for an appropriate distribution over roll calls. We stay in line with Eq. (1) by presuming that orderings are drawn independently from the uniform distribution on $\mathcal{S}_n$. However, we define value φ^p by

$$\varphi_i^p(v) = \frac{1}{n!} \sum_{\pi\in\mathcal{S}_n} \sum_{S\in 2^N} p(S) \cdot M(v, (\pi, S), i) \quad \text{for } i \in N \tag{5}$$

for an arbitrary probability distribution p on 2^N, i.e., requiring only $p(S) \geq 0$ for all $S \in 2^N$ and $\sum_{S\in 2^N} p(S) = 1$. Cooperation of players thus neither needs to be complete with $p(N) = 1$, nor independent with $p(S) = \prod_{i\in S} x_i \prod_{i\notin S}(1 - x_i)$ for $x_i \in [0, 1]$.

3. Results

Proposition 1. *Value φ^p is linear, efficient, and satisfies the null player property for every probability distribution p.*

Proof. The null player property is obvious from the definition. Linearity follows from recalling that $v^*(\mathcal{N}(\mathcal{R}, i)\cup\{i\}) - v^*(\mathcal{N}(\mathcal{R}, i)) = v(N \setminus \mathcal{N}(\mathcal{R}, i)) - v(N \setminus (\mathcal{N}(\mathcal{R}, i)\cup\{i\}))$. So φ^p is a linear combination of terms that are linear in v. For efficiency, first observe that

$$\begin{aligned} \sum_{i=1}^{n} M(v, \mathcal{R}, i) &= v(S) - v(\varnothing) + v^*(\bar{S}) - v^*(\varnothing) \\ &= v(N) - v(\varnothing) = v(N) \end{aligned} \tag{6}$$

for any $\mathcal{R} \in \mathcal{S}_n \times 2^N$ given the telescope sum behavior of $\sum_{i=1}^n M(v, \mathcal{R}, i)$. Second, $|\mathcal{S}_n| = n!$ and $\sum_{S\in 2^N} p(S) = 1$. □

Random variables $X_1, \dots, X_n$ are called *exchangeable* or *symmetrically dependent* if the $n!$ permutations $(X_{k_1}, \dots, X_{k_n})$ all have the same n-dimensional probability distribution (see, e.g., Feller, 1971, sec. 7.4). Applied to votes or binary cooperation choices, which φ^p treats as random variables, this is equivalent to $p(S) = p(S')$ whenever $|S| = |S'|$, i.e., the probability of a particular partition of N into cooperators S and non-cooperators $\bar{S}$ depends only on the number of (non-)cooperators rather than their identities.

Proposition 2. *If players' cooperation choices are exchangeable under p then φ^p is symmetric.*