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Stability analysis by dynamic dissipation inequalities: On merging frequency-domain techniques with time-domain conditions

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ABSTRACT

This paper provides a complete link between dissipative systems theory and a celebrated result on stability analysis with integral quadratic constraints (IQCs). This is achieved with a new stability characterization for feedback interconnections based on the notion of finite-horizon integral quadratic constraints with a terminal cost. As the main benefit, this opens up opportunities for guaranteeing constraints on the transient responses of trajectories in feedback loops within absolute stability theory. For systems affected by parametric uncertainties, we show how to generate tight robustly invariant ellipsoids on the basis of a classical frequency-domain stability test, with illustrations by a numerical example.

1. Introduction

The framework of integral quadratic constraints (IQCs) was developed in [1] and builds on the seminal contributions of Yakubovich [2] and Zames [3,4]. It provides a technique for analyzing the stability of an interconnection involving a finite dimensional linear time-invariant (LTI) system in feedback with another system without any particular description and called uncertainty in the sequel. The key idea is to capture the properties of the uncertainty through filtered energy relations of the output in response to inputs with finite energy. Mathematically, this is formalized by requiring the L_2-input–output pairs of the uncertainty to satisfy an IQC in the frequency domain defined by a so-called multiplier. Then stability of the interconnection is guaranteed if the LTI system satisfies a suitable frequency-domain inequality (FDI) involving the multiplier, which can be computationally verified by virtue of the Kalman–Yakubovich–Popov (KYP) lemma. Various papers (cf. [1,5] and references therein) give a detailed exposition of different uncertainties and their corresponding multiplier classes based on which the IQC theorem in [1] allows to generate practical computational robust stability and performance analysis tests. The stunningly wide impact of this framework also incorporates, among many others, the analysis of adaptive learning [6] or of optimization algorithms [7].

Another central notion in systems theory is dissipativity [8,9], which has been developed by Jan Willems with the explicit goal of arriving at a more fundamental understanding of the stability properties of feedback interconnections [8]. Roughly speaking, a system with a state–space description is said to be dissipative with respect to some supply rate if there exists a storage function for which a dissipation inequality is valid along all system trajectories; for quadratic supply rates, such dissipation inequalities can be also viewed as IQCs.

A large body of work has been devoted to analyzing the links between both frameworks. In particular, if the multipliers (supply rates) are non-dynamic and the two approaches involve so-called hard (finite-horizon) IQCs, the relation between the two worlds is well-established, as covered, e.g., by [10–14]; the classical small-gain, passivity or conic-sector theorems are prominent examples, with generalizations given in [15–19]. However, for the much more powerful dynamic multipliers in [1], the connection between the related so-called soft (infinite-horizon) IQCs and dissipativity theory has only been demonstrated for specialized cases in [20–25]. Relations of IQCs to Yakubovich's absolute stability framework and classical multiplier theory are discussed, e.g., in [26–32].

The purpose of this paper is to present a novel IQC theorem based on the notion of finite-horizon IQCs with a terminal cost. In generalizing [24,33,23], a first contribution is to show that the IQC theorem from [1] for general multipliers can be subsumed to our result, thus providing for the first time a tight link between the IQC framework and dissipativity theory. As argued in [20,34], such bridges permit to beneficially merge frequency-domain techniques with time-domain conditions, e.g., for the construction of local absolute stability criteria. This is illustrated by giving a novel loss-less Lyapunov proof for a well-known frequency domain robust stability test involving parametric uncertainties.

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The paper is structured as follows. In Section 2, we recall the main IQC theorem and formulate our new stability result based on dissipativity. Section 3 develops the technical ingredients that allow to link the IQC theorem with our encompassing result in Section 4. Finally, in Section 5 we illustrate the benefit of our framework over standard IQC theory for parametric uncertainties.

Notation. L_{2e}^n is the space of locally square integrable signals $x : [0,\infty) \to \mathbb{R}^n$ and $L_2^n = \{x \in L_{2e}^n \mid \|x\|^2 := \int_0^\infty x(t)^T x(t)\,dt < \infty\}$, while $\hat{x}$ denotes the Fourier transform of $x \in L_2^n$. For $T > 0$, $P_T : L_{2e}^n \to L_2^n$, $x \mapsto x_T$ with $x_T := x$ on $[0, T]$, $x_T := 0$ on (T, ∞) is the truncation operator. The system $S : L_{2e}^n \to L_{2e}^m$ is casual if $SP_T = P_T S P_T$ for all $T > 0$; S is bounded/stable if its L_2-gain is finite. For a transfer matrix G, we use $G^*(s) := G(-s)^T$ and $G = [A, B, C, D] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ means $G(s) = C(sI - A)^{-1}B + D$; further, $RL_\infty^{n\times m}$ ($RH_\infty^{n\times m}$) is the space of transfer matrices without poles on the extended imaginary axis $\mathbb{C}_=^\infty := i\mathbb{R} \cup \{\infty\}$ (in the closed right-half plane). Finally, $\mathrm{col}(u_1, \ldots, u_n)$ is the column vector with entries $u_1, \ldots, u_n$.

2. A novel IQC theorem

2.1. Recap of standard IQC theorem

For setting up the IQC framework, we consider the LTI system

$$\begin{array}{rcl} \dot{x} &=& Ax + Bw, \quad x(0) = 0, \\ z &=& Cx + Dw \end{array} \tag{1}$$

which defines the causal linear map $G : L_{2e}^{n_w} \to L_{2e}^{n_z}$ and the transfer matrix $G(s) = C(sI - A)^{-1}B + D$; all throughout the paper A is assumed to be Hurwitz. Given a system $\Delta : L_{2e}^{n_z} \to L_{2e}^{n_w}$ without any particular description, which is also called uncertainty and tacitly assumed to be causal and stable, we investigate the feedback interconnection

$$z = Gw + d \quad \text{and} \quad w = \Delta(z) \tag{2}$$

of G and Δ that is affected by the external disturbance $d \in L_{2e}^{n_z}$. The loop (2) is *well-posed* if, for any $d \in L_{2e}^{n_z}$, there exists a unique response $z \in L_{2e}^{n_z}$ that depends causally on d. The loop is *stable* if there exists some $\gamma \geq 0$ such that $\|z\| \leq \gamma\|d\|$ holds for all $d \in L_2^{n_z}$ and all responses of (2). We recall that (2) is well-posed and stable iff $(I - G\Delta) : L_{2e}^{n_z} \to L_{2e}^{n_z}$ has a causal bounded inverse.

Let us now cite the main theorem of [1] which involves a so-called multiplier Π, an essentially bounded Hermitian valued function on the imaginary axis.

Theorem 1. *The interconnection (2) is stable if there exists some $\epsilon > 0$ with*

$$\begin{pmatrix} G(i\omega) \\ I_{n_w} \end{pmatrix}^* \Pi(i\omega) \begin{pmatrix} G(i\omega) \\ I_{n_w} \end{pmatrix} \preceq -\epsilon I \ \text{ for almost all } \ \omega \in \mathbb{R}, \tag{3}$$

if, for all $\tau \in [0, 1]$, the integral quadratic constraints

$$\int_{-\infty}^{\infty} \begin{pmatrix} \hat{z}(i\omega) \\ \tau\widehat{\Delta(z)}(i\omega) \end{pmatrix}^* \Pi(i\omega) \begin{pmatrix} \hat{z}(i\omega) \\ \tau\widehat{\Delta(z)}(i\omega) \end{pmatrix} d\omega \geq 0 \ \text{ for all } \ z \in L_2^{n_z} \tag{4}$$

are satisfied, and if (2) is well-posed for $\tau\Delta$ replacing Δ.

In order to apply the KYP Lemma to (3), we work throughout this paper with rational multipliers. Following [1], these can be described, w.l.o.g., in terms of a (usually tall) stable outer factor Ψ and a real middle matrix M as

$$\Pi = \Psi^* M \Psi \text{ with } \Psi \in RH_\infty^{n_y \times (n_z + n_w)} \text{ and } M = M^T \in \mathbb{R}^{n_y \times n_y}. \tag{5}$$

We emphasize that many multiplier classes do admit a description (5) with some fixed Ψ and a variable M (see e.g. [1,5]). For the filter $\Psi = \begin{pmatrix} \Psi_1 & \Psi_2 \end{pmatrix}$ with a column partition according to $n_z + n_w$, we introduce the state–space description

$$\begin{array}{rcl} \dot{\xi} &=& A_\Psi \xi + B_{\Psi_1} z + B_{\Psi_2} w, \quad \xi(0) = 0, \\ y &=& C_\Psi \xi + D_{\Psi_1} z + D_{\Psi_2} w \end{array} \tag{6}$$

where A_Ψ is Hurwitz.

2.2. Main result

On the basis of (1) and (6) let us now introduce the natural realization

$$\begin{pmatrix} \Psi_1 & \Psi_2 \end{pmatrix} \begin{pmatrix} G \\ I \end{pmatrix} = \left[\begin{array}{cc|c} A_\Psi & B_{\Psi_1} C & B_{\Psi_1} D + B_{\Psi_2} \\ 0 & A & B \\ \hline C_\Psi & D_{\Psi_1} C & D_{\Psi_1} D + D_{\Psi_2} \end{array}\right] =: \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] \tag{7}$$

for the filtered transfer matrix of the inverse system graph. Since A_Ψ and A are Hurwitz, the same holds for $\mathcal{A}$. By the KYP-Lemma, (3) is equivalent to the existence of some $\mathcal{X} = \mathcal{X}^T$ that satisfies the KYP inequality

$$\mathscr{L}\left(\mathcal{X}, M, \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}\right) := \begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}^T \begin{pmatrix} 0 & \mathcal{X} & 0 \\ \mathcal{X} & 0 & 0 \\ 0 & 0 & M \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \prec 0; \tag{8}$$

note that the operator $\mathscr{L}$ is just introduced to save space. In the sequel we say that $\mathcal{X}$ certifies the FDI (3), or that $\mathcal{X}$ is a certificate thereof; whenever relevant we assume $\mathcal{X}$ to be partitioned as $\mathcal{A}$ in (7).

Now let (8) be valid. By Finsler's lemma, we can choose some $\gamma > 0$ with

$$\mathscr{L}\left(\mathcal{X}, \begin{pmatrix} M & 0 & 0 \\ 0 & \frac{1}{\gamma}I & 0 \\ 0 & 0 & -\gamma I \end{pmatrix}, \left(\begin{array}{cc|cc} A_\Psi & B_{\Psi_1} C & B_{\Psi_1} D + B_{\Psi_2} & B_{\Psi_1} \\ 0 & A & B & 0 \\ \hline C_\Psi & D_{\Psi_1} C & D_{\Psi_1} D + D_{\Psi_2} & D_{\Psi_1} \\ 0 & C & D & I_{n_z} \\ 0 & 0 & 0 & I_{n_z} \end{array}\right)\right) \prec 0. \tag{9}$$

This leads to a crucial dissipation inequality as follows. If $z = Gw + d$ is the response to any $w \in L_{2e}^{n_w}$, $d \in L_{2e}^{n_z}$ and if we let z, w drive the filter (6), we infer

$$\begin{aligned} \begin{pmatrix} \dot{\xi} \\ \dot{x} \end{pmatrix} &= \begin{pmatrix} A_\Psi & B_{\Psi_1} C \\ 0 & A \end{pmatrix} \begin{pmatrix} \xi \\ x \end{pmatrix} + \begin{pmatrix} B_{\Psi_1} D + B_{\Psi_2} & B_{\Psi_1} \\ B & 0 \end{pmatrix} \begin{pmatrix} w \\ d \end{pmatrix}, \\ \begin{pmatrix} \xi(0) \\ x(0) \end{pmatrix} &= 0, \\ \begin{pmatrix} y \\ z \\ d \end{pmatrix} &= \begin{pmatrix} C_\Psi & D_{\Psi_1} C \\ 0 & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ x \end{pmatrix} + \begin{pmatrix} D_{\Psi_1} D + D_{\Psi_2} & D_{\Psi_1} \\ D & I_{n_z} \\ 0 & I_{n_z} \end{pmatrix} \begin{pmatrix} w \\ d \end{pmatrix}. \end{aligned}$$

Therefore, with the combined state trajectory $\eta = \mathrm{col}(\xi, x)$, we can right- and left-multiply (9) by $\mathrm{col}(\eta, w, d)$ and its transpose to obtain

$$\frac{d}{dt}\eta(t)^T \mathcal{X}\eta(t) + y(t)^T M y(t) + \frac{1}{\gamma}\|z(t)\|^2 - \gamma\|d(t)\|^2 \leq 0$$
for almost all $t \geq 0$.

After integration we arrive at the following dissipation inequality for all $T > 0$:

$$\eta(T)^T \mathcal{X}\eta(T) + \int_0^T y(t)^T M y(t)\,dt + \int_0^T \frac{1}{\gamma}\|z(t)\|^2 - \gamma\|d(t)\|^2\,dt. \tag{10}$$

On the other hand, let us consider the IQC (4) with $\tau = 1$:

$$\int_{-\infty}^{\infty} \begin{pmatrix} \hat{z}(i\omega) \\ \widehat{\Delta(z)}(i\omega) \end{pmatrix}^* \Psi(i\omega)^* M \Psi(i\omega) \begin{pmatrix} \hat{z}(i\omega) \\ \widehat{\Delta(z)}(i\omega) \end{pmatrix} d\omega \geq 0 \tag{11}$$
for all $z \in L_2^{n_z}$.