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Structure preserving schemes for the continuum Kuramoto model: Phase transitions

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A B S T R A C T

The construction of numerical schemes for the Kuramoto model is challenging due to the structural properties of the system which are essential in order to capture the correct physical behavior, like the description of stationary states and phase transitions. Additional difficulties are represented by the high dimensionality of the problem in presence of multiple frequencies. In this paper, we develop numerical methods which are capable to preserve these structural properties of the Kuramoto equation in the presence of diffusion and to solve efficiently the multiple frequencies case. The novel schemes are then used to numerically investigate the phase transitions in the case of identical and nonidentical oscillators.

1. Introduction

Synchronization phenomena of large populations of weakly coupled oscillators are very common in natural systems, and it has been extensively studied in various scientific communities such as physics, biology, sociology, etc. [2,4,29,52]. Synchronization arises due to the adjustment of rhythms of self-sustained periodic oscillators weakly connected [2,47], and its rigorous mathematical treatment is pioneered by Winfree [53] and Kuramoto [4]. In [4,53], phase models for large weakly coupled oscillator systems were introduced, and the synchronized behavior of complex biological systems was shown to emerge from the competing mechanisms of intrinsic randomness and sinusoidal couplings. Since then, the Kuramoto model becomes a prototype model for synchronization phenomena and various extensions have been extensively explored in various scientific communities such as applied mathematics, engineering, control theory, physics, neuroscience, biology, and so on [28,47,52].

Given an ensemble of sinusoidally coupled nonlinear oscillators, which can be visualized as active rotors on the unit circle $\mathbb{S}^1$, let $z_j = e^{i\vartheta_j}$ be the position of the j-th rotor. Then, the dynamics of z_j is completely determined by that of its phase ϑ_j. Let us denote the phase and frequency of the j-th oscillator by ϑ_j and $\dot{\vartheta}_j$, respectively. Then, the phase dynamics of Kuramoto oscillators are governed by the following first-order ODE system [4]:

$$\dot{\vartheta}_i = \omega_i - \frac{K}{N}\sum_{j=1}^{N} \sin(\vartheta_i - \vartheta_j), \quad i = 1, \cdots, N, \quad t > 0, \tag{1.1}$$

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subject to initial data $\vartheta_i(0) =: \vartheta_{i0}$, $i = 1, \cdots, N$, where K is the uniform positive coupling strength, and ω_i denotes the natural phase-velocity (frequency) and is assumed to be a random variable extracted from the given distribution $g = g(\omega)$ satisfying

$$\int_{\mathbb{R}} g(\omega)\, d\omega = 1.$$

We notice that the first term and second term in the right hand side of the equation (1.1) represent the intrinsic randomness and the nonlinear attraction–repulsion coupling, respectively.

The system (1.1) has been extensively studied, and it still remains a popular subject in nonlinear dynamics and statistical physics. We refer the reader to [2] and references therein for general survey of the Kuramoto model and its variants. In [4], Kuramoto first observed a continuous phase transition in the continuum Kuramoto model ($N \to \infty$, see (1.2) below) with a symmetric distribution function $g(\omega)$ by introducing an order parameter r^∞ which measures the degree of the phase synchronization in the mean-field limit. More precisely, the order parameter is given by

$$r^N(t)e^{i\varphi_N(t)} := \frac{1}{N}\sum_{j=1}^{N} e^{i\vartheta_j(t)}, \quad r^\infty := \lim_{t\to\infty}\lim_{N\to\infty} r^N(t).$$

In fact, Kuramoto showed that the order parameter r^∞ as a function of the coupling strength K changes from zero (disordered state) to a non-zero value (ordered state) when the coupling strength K exceeds a critical value $K_c := 2/(\pi g(0))$, i.e., $r^\infty(K) = 0$ (incoherent state) for $K \in [0, K_c)$, $1 \geq r^\infty(K) > 0$ (coherent state) for $K > K_c$, and $r^\infty(K)$ increases with K. Later, it is also observed that the continuum Kuramoto model can exhibit continuous or discontinuous phase transition by taking into account different types of natural frequency distribution functions [11,12,45]. Here, continuous phase transitions refer to the continuity at $K = K_c$ of the order parameter $r^\infty(K)$. It is known to be continuous for the Gaussian distributed in frequency oscillators [52] and discontinuous for the uniformly distributed in frequency oscillators [45].

As the number of oscillators goes to infinity ($N \to \infty$), a continuum description of the system (1.1) can be rigorously derived by employing by now standard mean-field limit techniques for the Vlasov equation [19,21,40,42]. Let $\rho = \rho(\vartheta, \omega, t)$ be the probability density function of Kuramoto oscillators in $\vartheta \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ with a natural frequency ω extracted from a distribution function $g = g(\omega)$ at time t. Then the continuum Kuramoto equation is given by

$$\begin{aligned} &\partial_t \rho + \partial_\vartheta (u[\rho]\rho) = 0, \qquad (\vartheta, \omega) \in \mathbb{T}\times\mathbb{R}, \;\; t > 0, \\ &u[\rho](\vartheta, \omega, t) = \omega - K \int_{\mathbb{T}\times\mathbb{R}} \sin(\vartheta - \vartheta_*)\rho(\vartheta_*, \omega, t)g(\omega)\, d\vartheta_* d\omega, \end{aligned} \tag{1.2}$$

subject to the initial data:

$$\rho(\vartheta, \omega, 0) =: \rho_0(\vartheta, \omega), \qquad \int_{\mathbb{T}} \rho_0(\vartheta, \omega)\, d\vartheta = 1. \tag{1.3}$$

The order parameter r and the average phase φ associated to (1.2) are given as

$$r(t)e^{i\varphi(t)} = \int_{\mathbb{T}\times\mathbb{R}} e^{i\vartheta}\rho(\vartheta, \omega, t)g(\omega)\, d\vartheta d\omega,$$

leading to

$$r(t) = \int_{\mathbb{T}\times\mathbb{R}} \cos(\vartheta - \varphi(t))\rho(\vartheta, \omega, t)g(\omega)\, d\vartheta d\omega. \tag{1.4}$$

For the continuum equation (1.2), global existence and uniqueness of measure-valued solutions are studied in [21,40] and important qualitative properties such as the Landau damping towards the incoherent stationary states were analyzed in [30].

A very relevant issue from the application viewpoint is how stable these stationary states and phase transitions are by adding noise to the system [4,50]. The mean-field limit equation associated to (1.1) with standard Gaussian noise of strength $\sqrt{2D}$, with $D > 0$, is

$$\begin{aligned} &\partial_t \rho + \partial_\vartheta (u[\rho]\rho) = D\partial_\vartheta^2 \rho, \qquad (\vartheta, \omega) \in \mathbb{T}\times\mathbb{R}, \;\; t > 0, \\ &u[\rho](\vartheta, \omega, t) = \omega - K \int_{\mathbb{T}\times\mathbb{R}} \sin(\vartheta - \vartheta_*)\rho(\vartheta_*, \omega, t)g(\omega)\, d\vartheta_* d\omega, \end{aligned} \tag{1.5}$$