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Multi-level spectral deferred corrections scheme for the shallow water equations on the rotating sphere

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ABSTRACT

Efficient time integration schemes are necessary to capture the complex processes involved in atmospheric flows over long periods of time. In this work, we propose a high-order, implicit–explicit numerical scheme that combines Multi-Level Spectral Deferred Corrections (MLSDC) and the Spherical Harmonics (SH) transform to solve the wave-propagation problems arising from the shallow-water equations on the rotating sphere.
The iterative temporal integration is based on a sequence of corrections distributed on coupled space–time levels to perform a significant portion of the calculations on a coarse representation of the problem and hence to reduce the time-to-solution while preserving accuracy. In our scheme, referred to as MLSDC-SH, the spatial discretization plays a key role in the efficiency of MLSDC, since the SH basis allows for consistent transfer functions between space–time levels that preserve important physical properties of the solution.
We study the performance of the MLSDC-SH scheme with shallow-water test cases commonly used in numerical atmospheric modeling. We use this suite of test cases, which gradually adds more complexity to the nonlinear system of governing partial differential equations, to perform a detailed analysis of the accuracy of MLSDC-SH upon refinement in time. We illustrate the stability properties of MLSDC-SH and show that the proposed scheme achieves up to eighth-order convergence in time. Finally, we study the conditions in which MLSDC-SH achieves its theoretical speedup, and we show that it can significantly reduce the computational cost compared to single-level Spectral Deferred Corrections (SDC).

1. Introduction

The numerical modeling of global atmospheric processes presents a challenging application area requiring accurate time integration methods for the discretized governing partial differential equations. These complex processes operate on a wide range of time scales but often have to be simulated over long periods of time – up to a hundred years for long-term paleoclimate studies – which constitutes a challenge for the design of stable and efficient integration schemes. One strategy for creating more efficient temporal integration schemes for such systems is to employ a semi-implicit scheme that allows larger time steps to be taken than with explicit methods at a cost that is less than that of fully implicit methods [14].

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A second strategy is to use a parallel-in-time strategy to solve multiple time steps concurrently on multiple processors. Examples of parallel-in-time methods include Parareal [22], the Parallel Full Approximation Scheme in Space and Time, (PFASST, [9]), and MultiGrid Reduction in Time (MGRIT, [11]). In this work, we consider semi-implicit, iterative, multi-level temporal integration methods based on Spectral Deferred Corrections (SDC) that are easily extended to high-order and also serve as a first step toward constructing parallel-in-time integration methods for the atmospheric dynamics based on PFASST.

SDC methods are first presented in [8] and consist in applying a sequence of low-order corrections – referred to as sweeps – to a provisional solution in order to achieve high-order accuracy. Single-level SDC schemes have been applied to a wide range of problems, including reacting flow simulation [3,21], atmospheric modeling [19], particle motion in magnetic fields [40], and radiative transport modeling [6]. In [19], a fully implicit SDC scheme is combined with the Spectral Element Method (SEM) to solve the shallow-water equations on the rotating sphere. The authors demonstrate that the SDC method can take larger stable time steps than competing explicit schemes such as leapfrog, second-order Runge–Kutta methods, and implicit second-order Backward Differentiation Formula (BDF) method without loss of accuracy.

The approach considered here for atmospheric simulations builds on the work of [32], in which a Multi-Level Spectral Deferred Corrections (MLSDC) scheme is proposed to improve the efficiency of the SDC time integration process while preserving its high-order accuracy. MLSDC relies on the construction of coarse space–time representations – referred to as levels – of the problem under consideration. The calculations are then performed on this hierarchy of levels in a way that shifts a significant portion of the computational burden to the coarse levels. As in nonlinear multigrid methods, the space–time levels are coupled by the introduction of a Full Approximation Scheme (FAS) term in the collocation problems solved on coarse levels. With this multi-level approach, the iterative correction process requires fewer fine sweeps than the standard SDC scheme but still achieves fast convergence to the fixed point solution. Synthetic numerical examples demonstrate the efficiency and accuracy of the MLSDC approach.

The MLSDC approach is combined here with a spatial discretization based on the global Spherical Harmonics (SH) transform to solve the shallow-water equations on the rotating sphere. This study is relevant for practical applications since the SH transform is implemented in major forecasting systems such as the Integrated Forecast System (IFS) at the European Centre for Medium-Range Weather Forecasts (ECMWF, [36]) and the Global Spectral Model (GSM) at the Japan Meteorological Agency (JMA, [20]). Using a highly accurate method in space significantly reduces the spatial discretization errors and allows us to focus on the temporal integration. Our approach, referred to as MLSDC-SH, uses a temporal splitting in which only the stiff linear terms in the governing equations are treated implicitly, whereas less stiff terms are evaluated explicitly. Here, the word *stiff* is used to denote the terms that limit the time step size of fully explicit schemes. The temporal integration scheme retains the main features of the multi-level algorithm presented in [32], and takes full advantage of the structure of the spatial discretization to achieve efficiency. Specifically, we construct accurate interpolation and restriction functions between space–time levels by padding or truncating the spectral representation of the variables in the SH transform. In addition, the spherical harmonics combined with the implicit–explicit temporal splitting considered in this work circumvent the need for a global linear solver and rely on an efficient local solver for the implicit systems.

We illustrate the properties of MLSDC-SH using a widely used suite of shallow-water test cases [12,39]. We start the numerical study with a steady-state benchmark that highlights the connection between the magnitude of the spectral coefficients truncated during coarsening and the convergence rate of MLSDC-SH upon refinement in time. Then we proceed to more challenging unsteady test cases to show that MLSDC-SH is stable for large time steps and achieves up to eighth-order temporal convergence. Finally, we investigate the conditions in which the proposed scheme achieves its theoretical speedup and we demonstrate that MLSDC-SH can reduce the computational cost compared to single-level SDC schemes.

In the remainder of the paper, we first introduce the system of governing equations in Section 2. Then, we briefly review the fundamentals of the spatial discretization based on the global SH transform in Section 3. In Section 4, we describe the implicit–explicit temporal integration scheme, with an emphasis on the Multi-Level Spectral Deferred Corrections (MLSDC) scheme. Finally, in Section 5, we present numerical examples on the sphere demonstrating the efficiency and accuracy of our approach.

2. Governing equations

We consider the Shallow-Water Equations (SWE) on the rotating sphere. These equations capture the main horizontal effects present in the full atmospheric equations. Well-defined test cases are available – such as those considered in this work – that relate the SWE to some key features of the full atmospheric equations. Hence, they provide a simplified assessment of the properties of temporal and spatial discretizations for atmospheric simulations on the rotating sphere. We use the vorticity-divergence formulation [2,16] in which the prognostic variables $\boldsymbol{U} = [\Phi, \zeta, \delta]^T$ are respectively the potential, Φ, the vorticity, ζ, and the divergence, δ. Here, the vorticity and divergence state variables are used to overcome the singularities in the velocity field at the poles.

The system of governing partial differential equations is

$$\frac{\partial \Phi'}{\partial t} = -\nabla \cdot (\Phi' \boldsymbol{V}) - \bar{\Phi}\delta + \nu \nabla^2 \Phi', \tag{1}$$

$$\frac{\partial \zeta}{\partial t} = -\nabla \cdot (\zeta + f)\boldsymbol{V} + \nu \nabla^2 \zeta, \tag{2}$$