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Localized method of approximate particular solutions with polynomial basis functions

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ABSTRACT

The method of particular solutions (MPS) using polynomials as the basis functions has been successfully developed for solving a large class of partial differential equations. However, when a large number of collocation points are required, the above mentioned approach is not feasible since the resultant matrix is dense and ill-conditioned. This restriction is common for global methods. One of the alternative approaches is to employ the localized scheme in which only a small number of neighboring points are being used in the solution process. As a result, the resultant matrix of the localized method is sparse and thus can be solved efficiently.

In this paper, the localized method has been employed to extend the recently developed MPS using polynomial basis functions for solving large-scale science and engineering problems. Overall, the proposed approach is stable and highly accurate. Four numerical examples are presented to validate the proposed numerical method.

1. Introduction

The method of particular solutions (MPS) [4,5,19], which is a meshless method, is capable of solving a large class of partial differential equations in science and engineering. In the past, this method has employed the radial basis functions (RBFs) for the construction of particular solutions for the differential equations. The traditional MPS using RBFs possesses some challenges such as the determination for the shape parameter of the RBFs, finding the particular solutions for the differential operator of the governing equation, and the dense and ill-conditioned interpolation matrix [10,20]. Recently, Dangal et al. [6] has adopted polynomials as the basis functions to overcome these difficulties. Since this approach is a global method, it is still not suitable for solving large-scale problems. There are many traditional mesh-based localized numerical methods such as the finite element method [2,3,8] and the finite difference method [21] for solving large-scale problems. But these methods are quite tedious and cumbersome in terms of implementation particularly for problems defined in higher dimension with irregular domains. Therefore, localized meshless methods [7,11,15,16,18,22] are preferred over mesh-based methods because of their ability to solve problems defined on regular and irregular domains in higher dimensions. Traditionally, these localized meshless methods have utilized the radial basis functions in the solution process.

Most of the radial basis functions contain the so-called shape parameter. Finding the optimal shape parameter for the RBF is still an outstanding research problem. To avoid the difficulty of determining a good shape parameter of the RBFs, we propose the localized method of particular solutions using polynomial basis functions. The main attractions of the proposed approach are

- The closed-form particular solutions for the general linear partial differential equations with constant coefficients using polynomial basis functions are available.
- No shape parameter is required.
- The numerical accuracy is better than that with the method based on radial basis functions.

The paper is organized as follows. In Section 2, the closed-form particular solutions for the general linear partial differential equations with constant coefficients using polynomial basis functions are presented. In Section 3, the localized method of particular solutions using polynomial basis functions is formulated. In Section 4, four numerical examples are presented to show the efficacy of the proposed algorithm. A brief conclusion and future works are outlined in Section 5.

2. Polynomial particular solutions

In this section, we briefly review polynomial basis functions and show how to find the particular solutions for the general second order partial differential operators with constant coefficients. For simplicity, let us consider the 2D case. The traditional polynomial basis functions

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of degree not more than s is given as follows:

$$\begin{aligned}\mathbf{P}_s^2 &= \{x^{i-j}y^j \; : \; 0 \le j \le i, 0 \le i \le s\} \\ &= \{1, x, y, x^2, xy, y^2, \dots, x^s, x^{s-1}y, x^{s-2}y^2, \dots, y^s\}.\end{aligned} \tag{1}$$

Note that $\mathcal{N} = (s+1)(s+2)/2$ is the number of the polynomial basis functions in (1).

The key step of the MPS is the derivation of the closed-form particular solution of the given partial differential equation. In the past, the closed-form particular solutions for polynomial basis functions were largely restricted to Poisson and Helmholtz-type equation [9]. For the 2D Poisson equation with monomial right-hand side, a particular solution of

$$\Delta\phi = x^m y^n, m \ge 0, n \ge 0,$$

is given by [9]:

$$\phi(x,y) = \begin{cases} \displaystyle\sum_{k=1}^{[(n+2)/2]} (-1)^{k+1} \frac{m!n!x^{m+2k}y^{n-2k+2}}{(m+2k)!(n-2k+2)!}, & \text{for } m \ge n, \\ \displaystyle\sum_{k=1}^{[(m+2)/2]} (-1)^{k+1} \frac{m!n!x^{m-2k+2}y^{n+2k}}{(m-2k+2)!(n+2k)!}, & \text{for } m < n. \end{cases}$$

For 2D Helmholtz-type equation, a particular solution for

$$\Delta\phi + \varepsilon\lambda^2\phi = x^m y^n, m \ge 0, n \ge 0,$$

is given by [13]:

$$\phi(x,y) = \sum_{k=0}^{[m/2]} \sum_{\ell=0}^{[n/2]} \frac{\varepsilon(-\varepsilon)^{k+\ell}(k+\ell)!m!n!x^{m-2k}y^{n-2\ell}}{\lambda^{2k+2\ell+2}k!\ell!(m-2k)!(n-2\ell)!}.$$

Similar closed-form particular solution for monomial right-hand side for Poisson and Helmholtz-type equations for 3D are also available [9]. However, for more general types of partial differential equations containing convection terms, the closed-form particular solution is not available until recently. We briefly present the generalized polynomial particular solution in [6] as follows.

Consider a general second order linear partial differential equation in two variables with constant coefficients:

$$\alpha_1 \frac{\partial^2\phi}{\partial x^2} + \alpha_2 \frac{\partial^2\phi}{\partial x \partial y} + \alpha_3 \frac{\partial^2\phi}{\partial y^2} + \alpha_4 \frac{\partial\phi}{\partial x} + \alpha_5 \frac{\partial\phi}{\partial y} + \alpha_6\phi = x^m y^n, \tag{2}$$

where $\{\alpha_i\}_{i=1}^6$ are real constants, $\alpha_6 \ne 0$, and m and n are positive integers. Then the polynomial particular solution of (2) is given by [6]

$$\phi(x,y) = \frac{1}{\alpha_6} \sum_{k=0}^{m+n} \left(\frac{-1}{\alpha_6}\right)^k L^k(x^m y^n), \tag{3}$$

where

$$L = \alpha_1 \frac{\partial^2}{\partial x^2} + \alpha_2 \frac{\partial^2}{\partial x \partial y} + \alpha_3 \frac{\partial^2}{\partial y^2} + \alpha_4 \frac{\partial}{\partial x} + \alpha_5 \frac{\partial}{\partial y}.$$

The above mentioned solution procedure can be extended to the case $\alpha_6 = 0$. We refer readers to Reference [12] for further details.

The polynomial particular solution in (2) can be obtained and prestored prior to the numerical implementation for fast computation using MATLAB symbolic toolbox.

For the 3D case, similar results shown above can be obtained and $\mathcal{N} = (s+1)(s+2)(s+3)/6$.

3. LMPS formulation

We briefly explain the method of particular solutions (MPS) before we formulate the localized method of particular solutions (LMPS) using polynomial basis functions.

Let $f(\mathbf{x})$ and $g(\mathbf{x})$ be given functions. Consider the following boundary value problem

$$\mathcal{L}u(\mathbf{x}) = f(\mathbf{x}), \; \mathbf{x} \in \Omega, \tag{4}$$

$$\mathcal{B}u(\mathbf{x}) = g(\mathbf{x}), \; \mathbf{x} \in \Gamma, \tag{5}$$

where $\mathcal{L}$ is a linear elliptic partial differential operator, $\mathcal{B}$ is a boundary differential operator, and Ω is a closed and bounded domain with boundary Γ. By the method of particular solutions, we approximate the solution $u(\mathbf{x})$ of (4)–(5) by

$$\hat{u}(\mathbf{x}) = \sum_{i=0}^{s} \sum_{j=0}^{i} a_{ij} u_p^{ij}(\mathbf{x}), \tag{6}$$

where

$$\mathcal{L}u_p^{ij}(\mathbf{x}) = x^{i-j}y^j, \; 0 \le j \le i, 0 \le i \le s. \tag{7}$$

Let $\{\mathbf{x}_i\}_{i=1}^{N_i}$ be the interior point in Ω and $\{\mathbf{x}_i\}_{i=N_i+1}^{N_i+N_b}$ be the boundary point on Ω and $N = N_i + N_b$. Applying (6) to (4), we obtain

$$\sum_{i=0}^{s} \sum_{j=0}^{i} a_{ij} \mathcal{L}u_p^{ij}(\mathbf{x}_l) = f(\mathbf{x}_l), \; l = 1, 2, \dots, N_i. \tag{8}$$

From (7), the above equation becomes

$$\sum_{i=0}^{s} \sum_{j=0}^{i} a_{ij} x_l^{i-j} y_l^j = f(\mathbf{x}_l), \; l = 1, 2, \dots, N_i. \tag{9}$$

Applying (6) to the boundary condition (5), we get

$$\sum_{i=0}^{s} \sum_{j=0}^{i} a_{ij} \mathcal{B}u_p^{ij}(\mathbf{x}_l) = g(\mathbf{x}_l), \; l = N_i+1, N_i+2, \dots, N. \tag{10}$$

The system of Eqs (9) – (10) can be solved by the method of least square to obtain undetermined coeffiecients

$$\{a_{ij}\} = \{a_{00}, a_{10}, a_{11}, a_{20}, a_{21}, a_{22}, \cdots, a_{ss}\}.$$

The approximate solution $\hat{u}$ can be obtained from (6) by using undetermined coefficients $\{a_{ij}\}$.

In this paper, we extend the method of particular solutions (MPS) [6] to the localized method of particular solutions (LMPS) using polynomial basis functions. For each $\mathbf{x}_i$, we choose n nearest points $\mathbf{x}_k^{[i]}$, $k = 1, 2, \dots, n$, to form a local domain Ω_i which will more or less overlap with other local domains near it. In the LMPS, we approximate the solution at each point $\mathbf{x}_j^{[i]} \in \Omega_i$, $j = 1, 2, \dots, n$, by

$$u(\mathbf{x}_j^{[i]}) \simeq \hat{u}(\mathbf{x}_j^{[i]}) = \sum_{\kappa=0}^{\mathcal{N}} a_\kappa^{[i]} \Psi_\kappa(\mathbf{x}_j^{[i]}), \; j = 1, 2, \dots, n, \tag{11}$$

where $\mathcal{N}$ is the number of polynomial basis functions with order s and

$$\mathcal{L}\Psi_\kappa(\mathbf{x}_j^{[i]}) = x^m y^n, \; 0 \le m + n \le \mathcal{N}. \tag{12}$$

The system of equations in (11) can be written in the matrix form as follows:

$$\begin{bmatrix} \hat{u}(\mathbf{x}_1^{[i]}) \\ \hat{u}(\mathbf{x}_2^{[i]}) \\ \vdots \\ \hat{u}(\mathbf{x}_n^{[i]}) \end{bmatrix} = \begin{bmatrix} \Psi_1(\mathbf{x}_1^{[i]}) & \Psi_2(\mathbf{x}_1^{[i]}) & \Psi_3(\mathbf{x}_1^{[i]}) & \dots & \Psi_{\mathcal{N}}(\mathbf{x}_1^{[i]}) \\ \Psi_1(\mathbf{x}_2^{[i]}) & \Psi_2(\mathbf{x}_2^{[i]}) & \Psi_3(\mathbf{x}_2^{[i]}) & \dots & \Psi_{\mathcal{N}}(\mathbf{x}_2^{[i]}) \\ \vdots & & & & \\ \Psi_1(\mathbf{x}_n^{[i]}) & \Psi_2(\mathbf{x}_n^{[i]}) & \Psi_3(\mathbf{x}_n^{[i]}) & \dots & \Psi_{\mathcal{N}}(\mathbf{x}_n^{[i]}) \end{bmatrix} \begin{bmatrix} a_1^{[i]} \\ a_2^{[i]} \\ \vdots \\ a_{\mathcal{N}}^{[i]} \end{bmatrix}. \tag{13}$$

From (13), we have

$$\mathbf{a}^{[i]} = \mathbf{\Phi}_{n,s}^{-1} \hat{\mathbf{u}}^{[i]}, \tag{14}$$

where $\mathbf{\Phi}_{n,s}$ is the matrix on the right hand side of (13), and

$$\hat{\mathbf{u}}^{[i]} = \left[\hat{u}(\mathbf{x}_1^{[i]}), \hat{u}(\mathbf{x}_2^{[i]}), \dots, \hat{u}(\mathbf{x}_n^{[i]})\right]^T, \mathbf{a}^{[i]} = \left[a_1^{[i]}, a_2^{[i]}, \dots, a_{\mathcal{N}}^{[i]}\right]^T.$$

Then, applying (14) to (11), we obtain

$$\hat{u}(\mathbf{x}_j^{[i]}) = \Psi^{[i]}\mathbf{a}^{[i]} = \Psi_\kappa^{[i]} \mathbf{\Phi}_{n,s}^{-1} \hat{\mathbf{u}}^{[i]} = \Theta^{[i]} \hat{\mathbf{u}}^{[i]}, \tag{15}$$

where

$$\Theta^{[i]} = \Psi^{[i]} \mathbf{\Phi}_{n,s}^{-1},$$