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Constraints for geodesic network interpolation at a vertex ☆

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ABSTRACT

Consider n space curves $\widetilde{\mathbf{c}}_i(t)$ $i \in \mathbb{Z}_n$, meeting at a vertex. The geodesic network interpolation problem is to determine n surface pieces (patches) $\mathbf{x}_i(u,v)$ that are each internally C^2, surround the vertex and each interpolate curves $\widetilde{\mathbf{c}}_i$ and $\widetilde{\mathbf{c}}_{i+1}$ so that the curves are geodesics of the resulting surface. This paper proves that together three local constraints on the curves are necessary and sufficient for the existence of the surface patches $\mathbf{x}_i$: the binormal constraint, the geodesic crossing constraint and the vertex enclosure constraint. Additionally, the paper exhibits stronger geometric constraints, in terms of curvature and torsion, that imply the existence of a solution to the geodesic network interpolation problem.

1. Introduction

One approach to surface modeling is to create a network of curves and then determine a surface that interpolates this network. A variant is to interpolate curves so they serve as isoparametric geodesics of the resulting surface. A number of publications, listed below, observe that sewing together pieces of garment is simplified when their boundaries become straight or near-straight lines when flattened into the plane, i.e. are geodesic. A similar application is the initial layout of long fibers in composite material structures. Another group of publications trace their interest back to a measuring-tape-like device (shown in Sprynski et al. (2008)) that mimics a geodesic when placed onto a smooth physical surface to sample position and orientation. These publications outline the construction of surface pieces such that one or more of their iso-parameter curves are geodesics. Here we add to the understanding of surfaces with geodesic iso-parameter curves by characterizing when *smooth* surfaces can interpolate a *network* of geodesic curves.

It is well-known that the main challenge in curve network interpolation occurs where $n \neq 4$ patches enclose a common vertex. For example $\mathbf{C}^1$ interpolation by patches $\mathbf{x}_i$ is always possible if n is odd, but requires an additional constraint, the vertex enclosure constraint, when n is even (Du and Schmitt, 1988; Hermann et al., 2009; Peters, 1991). Higher-order smoothness yields yet more complex constraints (Hermann et al., 2012). This paper focuses on C^1 interpolation but adds the requirement that the curves be geodesics of the resulting surface. We exhibit the conditions on a geodesic network to allow for interpolation and smooth vertex enclosure.

The paper is organized as follows. After reviewing geodesics and formally defining geodesic smooth vertex network interpolation in Section 2, Section 3 derives a necessary and sufficient triple of constraints for geodesic smooth vertex

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network interpolation. Then in Section 4, we give a sufficient *geometric* constraint for the same and discuss the relations between geometric and geodesic smooth vertex network interpolation constraints. Section 5 focuses on the case $n = 4$.

1.1. Prior work

Wang et al. (2004) characterized surfaces with one prescribed geodesic corresponding to a parameter line and demonstrated their use in garment design. The surfaces are described as a linear combination of 'marching-scale functions' $u(r,t)$, $v(r,t)$ and $w(r,t)$ and the Frénet frame along the curve. The paper exhibits necessary and sufficient constraints for marching scale functions to satisfy both the geodesic and the isoparametric requirements. Kasap et al. (2008) generalized the approach to more general marching-scale functions and derived a sufficient constraint for the curve to be an isoparametric geodesic boundary. Analogous constraints were used to construct quadric surfaces (Sprynski et al., 2008), polynomial surfaces (Paluszny, 2008; Sánchez-Reyes and Dorado, 2008), developable surfaces (Li et al., 2011) and minimal surfaces (Li et al., 2013) including one or two boundary geodesics. We note that the constraints restrict the surface, not the curves.

By contrast, Farouki et al. (2009a) considered constraints on curve segments to admit surfaces. They identified two types of constraints that are sufficient and necessary for the existence of analytic surfaces that interpolate four boundary curves as geodesics; and constructed a four-sided interpolating Bézier patch (Farouki et al., 2009b) and a three-sided Coons patch (Farouki et al., 2010). Constraints for low degree Bézier and B-spline patch to interpolate a geodesic quadrilateral appeared in Huogen and Guozhao (2015a,b). None of these references considers a geodesic network.

2. Preliminaries

For a space curve $\widetilde{\mathbf{c}} : \mathbb{R} \to \mathbb{R}^3$, we denote vector-valued maps $\widetilde{\mathbf{t}}, \widetilde{\mathbf{b}}$ and $\widetilde{\mathbf{m}}$ and scalar-valued maps $\widetilde{\kappa}$ and $\widetilde{\tau}$.

$$\widetilde{\mathbf{t}} := \widetilde{\mathbf{c}}' \text{ tangent}, \quad \widetilde{\mathbf{b}} := \frac{\widetilde{\mathbf{t}} \times \widetilde{\mathbf{t}}'}{\|\widetilde{\mathbf{t}} \times \widetilde{\mathbf{t}}'\|} \text{ binormal}, \quad \widetilde{\mathbf{m}} := \frac{\widetilde{\mathbf{b}} \times \widetilde{\mathbf{t}}}{\|\widetilde{\mathbf{t}}\|} \text{ main normal},$$
$$\widetilde{\kappa} := \frac{\|\widetilde{\mathbf{t}} \times \widetilde{\mathbf{t}}'\|}{\|\widetilde{\mathbf{t}}\|^3} \text{ curvature}, \qquad \widetilde{\tau} := \frac{det(\widetilde{\mathbf{t}}, \widetilde{\mathbf{t}}', \widetilde{\mathbf{t}}'')}{\|\widetilde{\mathbf{t}} \times \widetilde{\mathbf{t}}'\|^2} \text{ torsion}.$$

We assume that all curves are regular and free of inflection i.e., $\widetilde{\mathbf{t}} \neq 0$, and $\widetilde{\mathbf{t}} \times \widetilde{\mathbf{t}}' \neq 0$, so that $\widetilde{\mathbf{b}}, \widetilde{\mathbf{m}}, \widetilde{\kappa}$ and $\widetilde{\tau}$ are well-defined. For any map, e.g. $\widetilde{\mathbf{b}}_i$, we abbreviate

$$\mathbf{b}_i := \widetilde{\mathbf{b}}_i(0),\ \mathbf{b}_i' := \widetilde{\mathbf{b}}_i'(0), \text{etc. except } \kappa_i := \sigma_i \widetilde{\kappa}_i(0), \quad \sigma_i := \operatorname{sgn}(\det(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_i')).$$

For a regular oriented surface $\mathbf{x}(u, v)$:

$$\partial_1 \mathbf{x} := \frac{\partial \mathbf{x}}{\partial u}, \qquad \partial_2 \mathbf{x} := \frac{\partial \mathbf{x}}{\partial v}, \text{ and the unit normal vector } \widetilde{\mathbf{n}} := \frac{\partial_1 \mathbf{x} \times \partial_2 \mathbf{x}}{\|\partial_1 \mathbf{x} \times \partial_2 \mathbf{x}\|}.$$

A curve $\widetilde{\mathbf{c}}$ without inflection point is a *geodesic* on the surface if and only if the main normal vector along the curve is parallel to the normal vector of the surface: $\widetilde{\mathbf{m}} \parallel \widetilde{\mathbf{n}}$.

Definition 1. (Smooth vertex network) Let $\widetilde{\mathbf{c}}_i : [0,1] \to \mathbb{R}^3$, $i \in \mathbb{Z}_n = \{1, 2, \ldots, n\}$ be a sequence of n curves in $\mathbb{R}^3$ and

$$j := \begin{cases} 1 & \text{if } i = n, \\ i+1 & \text{else} \end{cases}$$

so that $\widetilde{\mathbf{c}}_j$ is cyclically the next neighbor of $\widetilde{\mathbf{c}}_i$. The $\widetilde{\mathbf{c}}_i$ form a *smooth vertex network* if they meet at a common point $\mathbf{P}$ in a plane with oriented unit normal $\mathbf{n} := \frac{\mathbf{t}_1 \times \mathbf{t}_2}{\|\mathbf{t}_1 \times \mathbf{t}_2\|} \neq 0$ and successive tangents meet at angles $\psi_i := \measuredangle(\mathbf{t}_i, \mathbf{t}_j)$ (see Fig. 1) so that

$$\widetilde{\mathbf{c}}_i(0) = \mathbf{P}, \quad \widetilde{\mathbf{c}}_i'(0) =: \mathbf{t}_i,\ \mathbf{t}_i \perp \mathbf{n}, \quad 0 < \psi_i < \pi. \tag{1}$$

We then define geodesic network interpolation as follows.

Definition 2. (geodesic smooth vertex network interpolation) Given a smooth vertex network $\widetilde{\mathbf{c}}_i, i = 1, \ldots, n$ and twice differentiable patches $\mathbf{x}_i(u, v)$. The $\mathbf{x}_i$ form a *geodesic smooth vertex network interpolation* surface of $\widetilde{\mathbf{c}}_i$ if

$$\mathbf{x}_i(t, 0) = \widetilde{\mathbf{c}}_i(t), \quad \mathbf{x}_i(0, t) = \widetilde{\mathbf{c}}_j(t) \qquad \text{(boundary constraint)} \tag{2}$$
$$\widetilde{\mathbf{m}}_i(t) \parallel \widetilde{\mathbf{n}}_i(t, 0), \quad \widetilde{\mathbf{m}}_j(t) \parallel \widetilde{\mathbf{n}}_i(0, t). \qquad \text{(geodesic constraint)} \tag{3}$$