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Analysis of the stability and dispersion for a Riemannian acoustic wave equation

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A B S T R A C T

The construction of images of the Earth's interior using methods as reverse time migration (RTM) or full wave inversion (FWI) strongly depends on the numerical solution of the wave equation. A mathematical expression of the numerical stability and dispersion for a particular wave equation used must be known in order to avoid unbounded numbers of amplitudes. In case of the acoustic wave equation, the Courant–Friedrich–Lewy (CFL) condition is a necessary but is not a sufficient condition for convergence. Thus, we need to search other types of expression for stability condition. In seismic wave problems, the generalized *Riemannian wave equation* is used to model their propagation in domains with curved meshes which is suitable for zones with rugged topography. However, only a heuristic version of stability condition was reported in the literature for this equation. We derived an expression for stability condition and numerical dispersion analysis for the Riemannian acoustic wave equation in a two-dimensional medium and analyzed its implications in terms of computational cost.

1. Introduction

One of the most important features when simulating wave propagation into the earth is to design stable, accurate and efficient numerical methods which represent the propagation phenomena and generate images of the earths interior which represent its constitution, geometry, and layers distribution. Some of the methods that recently have been used to simulate acoustic wave propagation in different scenarios and with several boundary conditions are [8] where the boundary element method is used to solve 3D acoustic scattering; [13] used a hybrid methodology combining the finite element method and the wave based method to maximize the advantages and compensate the drawbacks of both numerical methods; [3] developed the singular boundary method for wave propagation analysis to eliminate singularities of the fundamental solutions and numerical evaluation of the singular integrals in the boundary element method. Nevertheless, the most straightforward method is the finite-difference time-domain (FDTD) approximation of the solution of the wave equation simulated on rectangular meshes. This method faces some difficulties to properly model the earth since the geological structures of the earth are described by rectangular meshes but the data acquisition can be made on irregular surfaces or rugged topography and this important aspect is not considered usually into the FDTD method.

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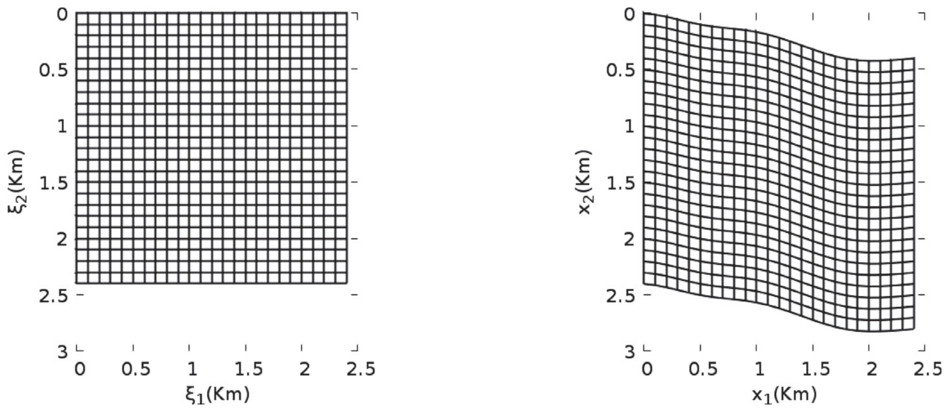

(a) Computational domain (b) Physical domain

Fig. 1. The transformation of coordinates given by equations $\boldsymbol{\phi}$ allows to go from (a) to (b).

One way to include the topography of the acquisition surface is to perform a conformal transformation from a domain with a curved upper surface to a rectangular Cartesian grid. The strategy of conformally transform the grids has been developed by many authors in curvilinear coordinates, for instance see [1,7,17], for the elastic wave equation and [18] for the acoustic wave equation. In these works the wave equation is transformed (a smooth coordinate transformation) to be implemented in a cartesian grid which conforms with a curved irregular grid, an orthogonality condition is possed and then some terms with mixed partial derivatives of the wavefield vanish, and the result is a wave equation with a transformed Laplacian operator. A FDTD scheme is used, and a stability condition is derived by Fourier analysis of the components of the wavefield. [1], established a second order stable finite difference scheme for the elastic wave equation in a curvilinear system, showing that the spatial operators in the method are self-adjoint for free-surface, nevertheless, the authors do not present a stability criteria. Recently in [20], the Von-Neumann method was applied for stability and numerical dispersion for a FD scheme for the diffusive-viscous wave equation, in that work the results obtained were compared with stability condition for the acoustic case and revealed that the stability condition is more restrictive for the diffusive-viscous case on which a smaller time step is required but numerical dispersion is also smaller than that in the acoustic case.

Nevertheless this approaches are dependent of the Euclidean structure to describe distances, angles between rays, overturning rays, the symmetries that the continuum may posses, among other issues. To properly describe the continuum in terms of its material symmetries and the tensor operators that are naturally defined on it, it is necessary to provide a general geometrical structure on which the propagation can be described in a general way. Riemannian geometry is one of the general structures which allows us to formally describe an elastic boby and to obtain the equations of motion after a deformation, see [9,12] proposed a Riemannian wave equation, as a pure eigenvalue equation by considering the Laplace–Beltrami operator acting on a pressure field. [16] included the topography geometry in the acoustic Riemannian wave equation and, [15] developed a FD scheme with conformal transformations of the domains. In this work, a stability condition is derived in a heuristic way taking as the starting point the CFL stability condition and transforming it by means of the chain rule. Formally, this is not a stability condition since it is not including the area and distances transformations, which are geometrical aspects intrinsically included in the metric tensor defined in a Riemannian manifold, and the time sampling resulting from this analysis to ensure stability of the FDTD scheme is quite far of being optimal.

Our aim is to apply the Von-Neumann method to obtain a stability criteria for a second order and fourth order FD scheme of the 2D Riemannian acoustic wave equation and compare it with the heuristic one used by Shragge (2014). For numerical comparison, we also perform a wave propagation using two different topography profiles: a Gaussian 2D profile and the Canadian Foothills profile, ([5]), a synthetic velocity model representing a zone in the British Columbia (Canada) that shows several geological complexities common in that region. This velocity model allows us to show the dependence of the stability criteria on the smoothness of the profile. Finally we analyze the numerical dispersion for the generalized wave equation and compare it with the Cartesian acoustic wave equation.

2. 2D Riemannian wave equation

In this section we make a review of the formulation of the Riemannian wave equation which agrees with the wave equations used by [12,14,15]. For a basic study on Riemannian manifolds the reader is referred to [10] and for the elastic formulation on Riemannian manifolds see [9].

Let $\boldsymbol{x} = [x_1, x_2]^T$ be coordinates of a curved physical domain on which we want to solve the wave equation (Fig. 1b) and, $\boldsymbol{\xi} = [\xi_1, \xi_2]^T$ a rectangular (regular) computational domain on which we actually compute the acoustic wavefield (Fig. 1a). Consider a function $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{\xi})$ that maps the computational domain onto the physical domain and the constant-density