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Length bounds for cycle bases of graphs ☆

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A B S T R A C T

The cycle space of a graph G is a vector space over $GF(2)$ which is formed by all Eulerian subgraphs of G with vector addition $X \oplus Y := (X \cup Y) \setminus (X \cap Y)$ and scalar multiplication $1 \cdot X = X, 0 \cdot X = \emptyset$. A base of this vector space is called a cycle base. A cycle base is used to examine the cyclic structure of a graph. The length of a cycle base is the number of its edges in the base. A minimum cycle base is that having the least number of edges. In this paper, we study the length bounds for cycle bases and the minimum cycle bases. A complete characterization is given for a 2-connected graph G with a cycle base of length $2|E(G)| - |V(G)|$ (it is a lower bound obtained by Leydold and Stadler (1998) [11]). In addition, we derive a sharp lower length bound for minimum cycle bases. As for upper bounds, Horton (1987) showed that the length of a minimum cycle base of a graph with n vertices is at most $3(n-1)(n-2)/2$. We improve the bound substantially for graphs on the projective plane where it is at most $\lfloor 13n/2 \rfloor - 9$.

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1. Introduction

In this paper, graphs are simple, unweighted and undirected. Most of our terminology is standard and can be found in [1,6,8,12,14,15].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. G_1 and G_2 are two subgraphs of G. We shall write $G_1 \setminus G_2$ for the subgraph of G induced by the edge set $E(G_1) \setminus E(G_2)$, and $G_1 \cup G_2$ for the subgraph of G induced by the edge set $E(G_1) \cup E(G_2)$. A *cycle* C is a connected subgraph such that every vertex in $V(C)$ has degree 2. The length $|C|$ of a cycle C is the number of its edges (in this case, we call C a $|C|$-cycle). A cycle C is *chordless* if C is an induced subgraph. The set ε of Eulerian subgraphs (unions of edge-disjoint cycles) forms a vector space over $GF(2)$ with vector addition $X \oplus Y := (X \cup Y) \setminus (X \cap Y)$ and scalar multiplication $1 \cdot X = X$, $0 \cdot X = \emptyset$, which is called the *cycle space* of G. Since the cycle space of a graph is the direct sum of the cycle spaces of its 2-connected components, we are concerned only with 2-connected graphs. It is well-known that the dimension of the cycle space ε is $\beta(G) = |E(G)| - |V(G)| + 1$, the *Betti number* of G. Further, any set of $\beta(G)$ linear independent Eulerian subgraphs forms a *cycle base*. The *length* of a set of walks (cycles) is the sum of lengths of its walks (cycles). The *length* of a cycle base $\mathcal{B}$, denoted by $l(\mathcal{B})$, is the total number of edges in its elements. If a cycle base has the shortest length, then it is called a *minimum cycle base*.

As applications, cycle base theory has many practical uses in fields such as electric circuit theory [4], structure analysis [3], biology and chemistry [5] and surface reconstruction [7], etc. In many of the above algorithms, the amount of works is dependent on the cycle base chosen. A cycle base with shorter cycles in it may speed up the algorithm. Thus one would like to be able to calculate the shortest cycle base possible, preferably quickly. Minimum cycle bases have been investigated by many people and much work has been done. Since Horton discovered the well-known algorithm for finding minimum cycle bases in [9], faster realizations have been discussed in [2,10,13].

In this paper, we consider the problem about the bounds on the length of a cycle base and a minimum cycle base of a graph. Leydold and Stadler [11] give a lower bound for the length of a cycle base $\mathcal{B}$ of a 2-connected graph G:

$$l(\mathcal{B}) \geq 2|E(G)| - |V(G)|.$$

At the same time, they [11] proved that when G is a Hamiltonian graph, there exists a cycle base $\mathcal{B}$ for which $l(\mathcal{B}) = 2|E(G)| - |V(G)|$ holds if and only if G is outerplanar (i.e. G can be embedded in the plane such that all vertices lie on the boundary of its exterior). Then what is about the case when G is not a Hamiltonian graph? For completeness, we characterize the structure of any graph with a minimum cycle base $\mathcal{B}$ of length $l(\mathcal{B}) = 2|E(G)| - |V(G)|$ as a decomposition theorem. On the other hand, we obtain a lower sharp bound on the length of the minimum cycle base of a 2-connected graph G. That is $3|E(G)| - 3|V(G)| + 3$.