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Recognizing the second derived subgroup of free groups

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ABSTRACT

The commutator subgroup of a free group is the subgroup consisting of elements contained in the kernel of every homomorphism from the free group to the integers. We give a similar characterization of the second derived subgroup of a free group. Specifically, we show that the second derived subgroup of a free group is equal to the intersection of the kernels of all homomorphisms into a solvable deficiency 1 group.

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1. Introduction and main result

In a free group, the commutator subgroup is the set of all words such that the exponent sum of every element of the free generating set is zero. Expressed differently, the commutator subgroup is the intersection of all kernels of homomorphisms from the free group to the integers. Here we prove a similar characterization for the second derived subgroup by considering the intersection of kernels of homomorphisms of a free group into a fixed solvable deficiency 1 group.

Theorem 1. *Let F be a free group and B any deficiency 1, solvable group which is not virtually abelian. Then*

$$\bigcap_{\phi \in \hom(F,B)} \ker(\phi) = F'',$$

where F'' is the second derived subgroup of F. Thus the evaluation map e gives rise to an exact sequence

$$1 \to F'' \to F \xrightarrow{e} B^{\hom(F,B)}.$$

Recall that the deficiency of a group H is the largest number n such that H has a presentation in which the number of generators exceeds the number of relations by n. Since the second commutator subgroup of a free group has infinite index in the commutator subgroup, it is necessary to consider homomorphisms into a fixed solvable deficiency 1 group that is not virtually abelian.

We shall derive Theorem 1 from the following:

Theorem 2. *Let G be a free metabelian group and B any solvable deficiency 1 group which is not virtually abelian. Then*

$$\bigcap_{\phi \in \hom(G,B)} \ker(\phi) = 1.$$

Expressed differently, the evaluation $e : G \to B^{\hom(G,B)}$ is a monomorphism.

2. Proofs

Notation. For sets A and B, we will denote the set of functions from A to B by B^A and, when A and B are groups, the set of homomorphisms from A to B by $\hom(A, B)$. When A and B are groups, B^A carries a natural group structure as follows. For ϕ and ψ in B^A, the product is the map $\phi\psi$ given by $\phi\psi(a) = \phi(a)\psi(a)$ for $a \in A$.

For elements x, y in a group G, let $x^y = yxy^{-1}$ denote the conjugation of x by y and $[x, y] = xyx^{-1}y^{-1}$ denote the commutator of x and y. We will denote the commutator subgroup of G by G' and the second commutator subgroup by G''.