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Group codes of dimension 2 and 3 are abelian

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ABSTRACT

Let F be a finite field and let G be a finite group. We show that if $\mathcal{C}$ is a G-code over F with $\dim_F(\mathcal{C}) \leq 3$ then $\mathcal{C}$ is an abelian group code. Since there exist non-abelian group codes of dimension 4 when char $F > 2$ (see the examples in [1]), we conclude that the smallest dimension of a non-abelian group code over a finite field is 4.

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0. Introduction

All groups and fields considered in this paper are supposed to be finite. Let F be a field and let G be a group. Following [2] we say that a linear code $\mathcal{C}$ over F is a (left) G-code if its length is equal to $n = |G|$ and there exists a one-to-one mapping $\nu : \{1, \ldots, n\} \to G$ such that

$$\left\{ \sum_{i=1}^{n} a_i \nu(i) : (a_1, \ldots, a_n) \in \mathcal{C} \right\}$$

is a (left) ideal in FG. We will also say that this (left) ideal is *permutation equivalent* to the code $\mathcal{C}$.

A code $\mathcal{C}$ is called an (*abelian*) *group code* if there exists an (abelian) group G such that $\mathcal{C}$ is a G-code.

It was shown in [2] that any one-dimensional group code over a field F is an abelian group code (moreover it is a C-code for a cyclic group C). It seems natural to ask: what is the lowest dimension of a non-abelian group code?

Since examples of non-abelian group codes of dimension 4 are known [1], a full answer to the above question is given in the main result of this paper.

Theorem 1. *Let $\mathcal{C}$ be a G-code over a finite field F for a finite group G. If* $\dim_F(\mathcal{C}) \leq 3$ *then $\mathcal{C}$ is an abelian group code.*

The paper is organized as follows. In section 1 we introduce some necessary notation and some auxiliary results are proved. In section 2 we prove Theorem 1.

1. Preliminaries

Let F be a field. We denote its multiplicative group by F^*. Let $M_{n,k}(F)$ be the vector space of $n \times k$ matrices over F, and let $M_n(F)$ be the algebra of all $n \times n$-matrices over F for any integers $n, k \geq 1$. We will use the notation $\mathrm{GL}_n(F)$, $\mathrm{D}_n(F)$, $\mathrm{T}_n(F)$ and $\mathrm{UT}_n(F)$ respectively for the group of all invertible $n \times n$-matrices, all invertible diagonal $n \times n$-matrices, the group of all invertible upper triangular $n \times n$-matrices and the group of all upper unitriangular $n \times n$-matrices, i.e. upper triangular matrices with diagonal elements equal to 1, over the field F.

Let us write $A \leq B$ to express that A is a subgroup of the group B, while $A \lhd B$ means that A is a normal subgroup in B. $Z(G)$ and $Z(R)$ will denote the center of the group G and of the ring R, respectively. We denote, for short, the set $\{m, m+1, \ldots, n\}$ by $\overline{m, n}$ for any integers $m \leq n$.

We recall the best known sufficient condition for all G-codes to be abelian.