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Combinatorial methods for invariance and safety of hybrid systems[☆]

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ABSTRACT

Inspired by Switching Systems and Automata theory, we investigate how combinatorial analysis techniques can be performed on a hybrid automaton in order to enhance its safety or invariance analysis. We focus on the particular case of *Constrained Switching Systems*, that is, hybrid automata with linear dynamics and no guards. We follow two opposite approaches, each with unique benefits: First, we construct invariant sets via the *'Reduced'* system, induced by a smaller graph which consists of the essential nodes, called the *unavoidable nodes*. The computational amelioration of working with a smaller, and in certain cases the minimum necessary number of nodes, is significant. Second, we exploit graph liftings, in particular the *Iterated Dynamics Lift* (T-Lift) and the *Path-Dependent Lift* (P-Lift). For the former case, we show that invariant sets can be computed in a fraction of the iterations compared to the non-lifted case, while we show how the latter can be utilized to compute non-convex approximations of invariant sets of a controlled complexity.

We also revisit well studied problems, highlighting the potential benefits of the approach. In particular, we apply our framework to (i) invariant sets computations for systems with dwell-time restrictions, (ii) fast computations of the maximal invariant set for uncertain linear systems and (iii) non-convex approximations of the minimal invariant set for arbitrary switching linear systems.

1. Introduction

Discrete-time linear switching systems consist of a finite collection of dynamics, called modes, which are allowed to switch at each time instant, according to a set of rules (see Eqs. (1)–(6) for a precise description). They constitute a particularly interesting and important family of hybrid systems Goebel, Sanfelice, & Teel (2012); Jungers (2009); Liberzon (2003) and Shorten, Wirth, Mason, Wulff, & King (2007). Apart from their simplicity, their ability to capture particular hybrid phenomena (Dehghan & Ong, 2012b; Donkers, Heemels, van den Wouw, & Hetel, 2011; Hernandez-Mejias, Sala, Arino, & Querol, 2015; Zhang, Zhuang, & Braatz, 2016) and approximate arbitrarily well nonlinear dynamics (Girard & Pappas, 2011) makes them a central model in the class of hybrid systems. Thus, it is not surprising that switching systems have been the subject of huge research efforts with existing techniques arguably more powerful than the ones targeted to general hybrid systems. Our goal in this paper is to push further the boundary of application of these techniques, by combining them with combinatorial techniques from graph and automata theory. As a first step, we tackle an intermediate family of systems, known as *constrained switching systems* (Athanasopoulos & Lazar, 2014; Dai, 2012; Philippe, Essick, Dullerud, & Jungers, 2015; Wang, Roohi, Dullerud, & Viswanathan, 2017). These systems are more general than classical switching systems in that they have their switching signals restricted by a labeled directed graph, namely the switching constraints graph. For example, in Fig. 1, the system switches between the modes 1 and 2 and an admissible switching sequence is the one that can be realized by a path in the directed graph $\mathcal{G}_1$.

Recently, multi-sets have been introduced in order to analyze invariance properties of constrained switching systems (Athanasopoulos, Smpoukis, & Jungers, 2017; Blanchini & Miani, 2008; De Santis, Di Benedetto, & Berardi, 2004; Philippe et al., 2015). A *multi-set*, is a collection of sets, one per node of the graph that defines the switching constraints. When a multi-set is invariant, the system trajectories that start from within this multi-set are always confined in one of its members. In this article we establish new, efficient, invariant (multi-)set constructions by exploiting the topological properties of the switching constraints graph. We highlight that the notion of multi-set is useful, beyond its proper

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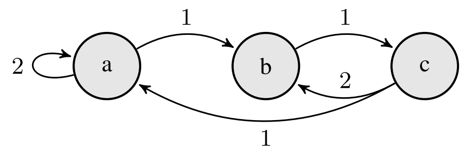

Fig. 1. A switching constraints graph $\mathcal{G}_1$ for a system with two modes. For example, the sequence 21121 is admissible whereas 212122 is not.

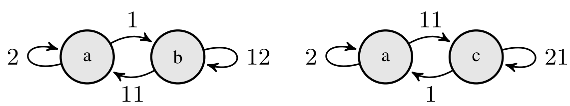

Fig. 2. Two possible reductions of $\mathcal{G}_1$ (Fig. 1). We observe that the sequence of labels appearing in any infinite length path of $\mathcal{G}_1$ can be generated by either graph.

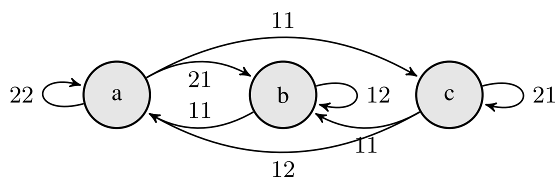

Fig. 3. The T-lifted graph of $\mathcal{G}_1$ of Fig. 1, $T = 2$. There are as many edges as admissible switching sequences of length 2 in $\mathcal{G}_1$.

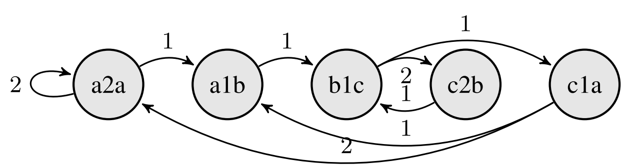

Fig. 4. The P-lifted graph of $\mathcal{G}_1$ of Fig. 1 using the Path-Dependent Lift, $P = 1$. The graph has as many nodes as different walks of length 1 in $\mathcal{G}_1$.

physical meaning, for improving the state of the art in classical problems on simpler models, like LTI systems, or arbitrary switching systems. We adopt two opposite and complementary approaches, one reducing and the other increasing the size of the graph.

The first direction borrows the concept of *unavoidability* of a set of nodes, a notion used in Computer Science, e.g., Lothaire (2002, Proposition 1.6.7). Roughly, by keeping only a subset of 'important' nodes we are able to show that we can work with a reduced graph, and consequently a reduced system, and associate explicitly invariance properties of the reduced system with the original one, leading to efficient algorithmic constructions. See for example two possible reduced graphs of $\mathcal{G}_1$ in Fig. 2.

The second direction considers the lifting of the switching constraints graph, a classical idea in switching systems analysis, e.g., Bliman and Ferrari-Trecate (2003), Lee and Dullerud (2006) and Philippe et al. (2015). Firstly, we consider the *Iterated Dynamics Lifted graph (abbr. T-lifted graph)*, which captures the switching constraints for the iterated dynamics of the systems, see, e.g., Fig. 3 for the 2-lift of $\mathcal{G}_1$ of Fig. 1. We exploit this construction to improve existing invariant multi-set computation algorithms by reducing the number of iterations required.

Secondly, we explore the *Path-Dependent Lifted graph (abbr. P-lifted graph)*, see, e.g., Fig. 4 for the 1-lifted graph of $\mathcal{G}_1$, in forward reachability computations. This choice enables us to establish algorithms for non-convex approximations of invariant multi-sets described by a union of a *prespecified* number of convex sets.

Together with the theoretical contributions, we revisit three problems of set invariance in control. In particular, we consider systems under *dwell-time specifications* (Dehghan & Ong, 2012a, b; Liberzon, 2003; Zhang et al., 2016). We compute, to the best of our knowledge for the first time, the minimal invariant multi-set and its approximations, via a Reduced graph consisting of the minimum number of nodes. Moreover, we compute the maximal invariant set for *uncertain linear systems* faster compared to the standard backward reachability algorithm, see e.g., Blanchini and Miani (2008). Last, we propose a new method to compute *non-convex approximations* of the minimal invariant set for switching systems (Artstein & Rakovic, 2008; Kolmanovsky & Gilbert, 1998; Rakovic, Kerrigan, Kouramas, & Mayne, 2005; Rakovic, Kouramas, Kerrigan, & Mayne, 2005).

Notation: The ball of radius α of an arbitrary norm is $\mathbb{B}(\alpha)$ and of the infinity norm is $\mathbb{B}_\infty(\alpha)$. The Minkowski sum of two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ is $\mathcal{S}_1 \oplus \mathcal{S}_2$. A C-set $\mathcal{S} \subset \mathbb{R}^n$ is a convex compact set which contains the origin in its interior (Blanchini, 1999). The cardinality of a set $\mathcal{V}$ is denoted by $|\mathcal{V}|$. Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$, or $\mathcal{G}$, be a labeled directed graph with a set of nodes $\mathcal{V}$ and a set of edges $\mathcal{E}$. The set of sequences of labels appearing in a path from a node $s \in \mathcal{V}$ to a node $d \in \mathcal{V}$ is denoted by $\sigma(s, d)$. The set of sequences of nodes appearing in a walk from $s \in \mathcal{V}$ to $d \in \mathcal{V}$ is $m(s, d)$. We denote the 1-norm of a vector x with $\|x\|_1$, and the vector with elements equal to one with $\mathbf{1}$. The convex hull of a set $\mathcal{S} \subset \mathbb{R}^n$ is denoted by $\mathrm{conv}(\mathcal{S})$.

2. Preliminaries

We consider a set of matrices $\mathcal{A} := \{A_1, \ldots, A_N\} \subset \mathbb{R}^{n\times n}$ and disturbance sets $\mathbb{W} = \{\mathcal{W}_1, \ldots, \mathcal{W}_N\}$, $\mathcal{W}_i \subset \mathbb{R}^n$. We consider the sets of nodes and edges $\mathcal{V} := \{1, 2, \ldots, M\}$ and $\mathcal{E} = \{(s, d, \sigma) : s \in \mathcal{V}, d \in \mathcal{V}, \sigma \in \{1, \ldots, N\}\}$. We denote the corresponding graph by $\mathcal{G}(\mathcal{V}, \mathcal{E})$, or $\mathcal{G}$. The set of outgoing nodes of a node $s \in \mathcal{V}$ is $\mathrm{Outgoing}(s, \mathcal{G}) := \{d \in \mathcal{V} : (\exists \sigma \in \{1, \ldots, N\} : (s, d, \sigma) \in \mathcal{E})\}$. Finally, we consider constraint sets $\mathcal{X}_i \subset \mathbb{R}^n$, $i \in \{1, \ldots, M\}$.

Formally, the systems we study are described by the following set of relations

$$x(t+1) = A_{\sigma(t)}x(t) + w(t), \tag{1}$$

$$z(t+1) \in \mathrm{Outgoing}(z(t), \mathcal{G}(\mathcal{V}, \mathcal{E})), \tag{2}$$

$$w(t) \in \mathcal{W}_{\sigma(t)}, \tag{3}$$

$$(x(0), z(0)) \in \mathbb{R}^n \times \mathcal{V}, \tag{4}$$

subject to the constraints

$$(z(t), z(t+1), \sigma(t)) \in \mathcal{E}, \tag{5}$$

$$x(t) \in \mathcal{X}_{z(t)}, \tag{6}$$

for all $t \geq 0$. We underline that the switching signal $\sigma(t)$ depends on the discrete variable $z(t)$ at each time instant, however for notational convenience we make a slight abuse and write $\sigma(t)$ instead of $\sigma(z(t))$. We note the system (1)–(6) is defined in the hybrid state space[1] $[x^\top \;\; z]^\top \in \mathbb{R}^n \times \mathcal{V}$. We call *nominal* the disturbance-free system, i.e., the system $x(t+1) = A_{\sigma(t)}x(t)$ together with (2), (4)–(6). The stability of the nominal system is characterized by the *constrained joint spectral radius* (Dai, 2012) $\check{\rho}(\mathcal{A}, \mathcal{G}) = \lim_{t\to\infty} \check{\rho}_t(\mathcal{A}, \mathcal{G})$, where

$$\check{\rho}_t(\mathcal{A}, \mathcal{G}) := \max\{\|A_{\sigma(t-1)} \cdots A_{\sigma(0)}\|^{1/t} : z(0) \in \mathcal{V},$$
$$z(t) \text{ satisfies } (2), \sigma(t) \text{ satisfies } (5), t = 0, \ldots, t-1\}.$$

The nominal system is asymptotically stable if and only if $\check{\rho}(\mathcal{A}, \mathcal{G}) < 1$ (Dai, 2012, Corollary 2.8). We consider the following assumptions.

Assumption 1. The constraint and disturbance sets $\mathcal{X}_i \subset \mathbb{R}^n$, $i = 1, \ldots, M$ and $\mathcal{W}_i$, $i = 1, \ldots, N$, are C-sets.

[1] Indeed, from (2), (4) it follows that $z(t) \in \mathcal{V}$, for all $t \geq 0$.