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Brief paper

Low complexity constrained control using higher degree Lyapunov functions[☆]

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ABSTRACT

Explicit Model Predictive Control often has a complex solution in terms of the number of regions required to define the solution and the corresponding memory requirement to represent the solution in the online implementation. An alternative approach to constrained control is based on the use of controlled contractive sets. However, polytopic controlled contractive sets may themselves be relatively complex, leading to a complex explicit solution, and the polytopic structure can limit the size of the controlled contractive set. This paper develops a method to obtain a larger controlled contractive set by allowing higher order functions in the definition of the contractive set, and explores the use of such higher-order contractive sets in controller design leading to a low complexity explicit control formulation.

1. Introduction

The ability to capture operational constraints is of vital importance in controller design for real-life applications. It is reasonable to state that the ability to handle constraints in a transparent way is what sets the industrially very successful Model Predictive Control (MPC) (Qin & Badgwell, 2003) apart from the theoretically elegant – but less industrially successful – LQG control. Standard MPC solves an optimization problem online, but due to the computational complexity of MPC it is limited to the systems which are not safety critical (due to the use of complex and thus error prone optimization software), have sufficiently slow dynamics, and/or can afford high performance computational hardware (Hovd, Olaru, & Bitsoris, 2014). Explicit MPC (Bemporad, Morari, Dua, & Pistikopoulos, 2002) to some degree resolves this problem and allows the use of low-complexity computing code in the online implementation. Unfortunately, the explicit solution to standard MPC problems often has a highly complex solution, and even in cases when the explicit solution can be found in acceptable time the implementation of the solution on the online control hardware may require excessive memory. Low complexity constrained control with modest computational complexity, small memory requirements and simple, thus verifiable code in the online implementation is therefore desired.

One approach to such low complexity constrained control is based on the use of a controlled contractive set. The complexity of the solution will then depend on the complexity of the contractive set. Therefore, obtaining a controlled contractive set of low complexity is essential for this approach to formulate low complexity explicit constrained control. A maximal polyhedral controlled contractive set with a given contraction factor can be obtained by the iterative procedure described in Dorea and Hennet (1999). However, the complexity of the contractive set thus obtained may be very high. A non-iterative procedure for obtaining a contractive set of low complexity is proposed in Hovd et al. (2014). The approach is not applicable to systems with identical modes in series (corresponding to a non-diagonalizable A-matrix in the system's state space representation). Furthermore, the contractive set obtained in Hovd et al. (2014) is of fixed complexity, which does not allow trading off the complexity against the size of the contractive set. An optimization based technique has been proposed in Munir, Hovd, Sandou, and Olaru (2016) which allows the trading off complexity versus the size of the set. A solution to the optimization problem in Munir et al. (2016) not only reduces the on-line computational complexity of the resulting constrained control, but also ensures significant reduction in the memory required to store the explicit solutions. However the method explained in Munir et al. (2016) is highly non-convex, which makes it difficult to use for finding sufficiently large contractive sets for higher dimensional systems. Alternatively, ellipsoidal contractive sets with corresponding linear control laws can be computed, but the measure of these sets is

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limited by the linear structure of the control law and the inherent conservatism of the corresponding quadratic Lyapunov function. This paper proposes a method to obtain an enlarged contractive set by defining the contractive set using a function of variable degree (a degree which is assumed to be greater or equal to 2, thus including the quadratic forms as a particular case), and also allowing for higher order control laws. Note that the function defining the controlled contractive set can be interpreted as control Lyapunov function for the closed loop system.

In Section 2, the controller design using controlled contractive sets is presented, along with the formulation for finding the largest ellipsoidal controlled contractive set fulfilling state and input constraints. Section 3 describes a controller design which leads to the determination of larger contractive sets. The controller design is inspired by the results in Oishi (2012), but unlike the respective work, the controller will be defined using only two regions. The method described in Section 3 is applied to illustrative examples and the results are described in Section 4, which is followed by a discussion and conclusions in Section 5.

2. Contractive sets

Consider the constrained control of the linear discrete time system:

$$x_{k+1} = Ax_k + Bu_k \tag{1}$$

with $x_k \in \Re^{n_x}$, $u_k \in \Re^{n_u}$ representing the current state and input, respectively, while x_{k+1} is the next time step state. The system is subject to input constraints $\mathcal{U} = \{u_k | H_u u_k \leq \mathbf{1}\}$, with $H_u \in \Re^{n_{pu} \times n_u}$.

Definition 1. Given a function $V : \Re^{n_x} \to \Re$, the level set of $V(x)$ for a scalar α is the set $S_\alpha = \{x | V(x) \leq \alpha\}$.

Proposition 1. *Consider a function $V(x) : \Re^{n_x} \to \Re$ satisfying the following properties:*

A1 positive definite, with $V(0) = 0$,
A2 continuous,
A3 radially unbounded, i.e., $V(x) \to \infty$ as $\|x\| \to \infty$.

Then

(1) All level sets S_α exist and are bounded for all $0 \leq \alpha < \infty$.
(2) If $\beta < \alpha$, $S_\beta \subset S_\alpha$.

Proof. From A1 it follows that the level sets $S_\alpha = \emptyset$ if $\alpha < 0$. Claim (1) follows directly from *A1*, *A2* and *A3*. For claim (2) we note that $S_\beta \subseteq S_\alpha$ is a consequence of Definition 1. Next, consider two points x_1 and x_2 with $V(x_1) = V(x_2) + \delta$ for some $\delta > 0$. Then by applying the Mean Value Theorem, continuity of $V(x)$ implies that the points x_1 and x_2 must be separated by some nonzero distance. Hence, we get strict inclusion, $S_\beta \subset S_\alpha$ if $\beta < \alpha$. □

Definition 2. Consider a continuous and radially unbounded function $V : \Re^{n_x} \to \Re_{\geq 0}$. A level set S_α is controlled γ-contractive with respect to (1) for a given $\gamma \in (0, 1)$, if $\forall x_k \in S_\alpha$, $\exists u_k \in \mathcal{U}$ such that $x_{k+1} \in S_{\gamma\alpha}$.

The functions $V(x)$ fulfilling the assumptions of Proposition 1 are natural ingredients in control designs enforcing contractiveness properties, as for example in the low complexity optimization based formulation

$$\min_{u_k, x_{k+1}} \frac{1}{2} x_{k+1}^T Q x_{k+1} + \frac{1}{2} u_k^T R u_k$$

subject to (2a)

$$x_{k+1} = Ax_k + Bu_k \tag{2b}$$

$$H_u u_k \leq \mathbf{1} \tag{2c}$$

$$V(x_{k+1}) \leq \gamma V(x_k) \tag{2d}$$

where Q and R represent the state and input weights.

Consider next the bounded state constraints $x_k \in \mathcal{X}$ with $\mathcal{X} = \{x_k | H_x x_k \leq \mathbf{1}\}$ where $H_x \in \Re^{p_x \times n_x}$.

Proposition 2. *Let $V(x)$ be a function fulfilling assumptions $A1 - A3$ of Proposition 1, and let $V(x) = \alpha$, Then, if*

(1) the corresponding level set S_α is controlled γ-contractive, and
(2) $S_\alpha \subseteq \mathcal{X}$

the control action obtained as a solution of (2) guarantees an exponentially stability of the closed loop which in addition fulfills input and state constraints over S_α.

Proof. Follows directly from Proposition 1 and Definition 2. □

As a result of Proposition 2, the function $V(x)$ is a Lyapunov function for the system (1) inside the set $S_{\bar{\alpha}}$, where $\bar{\alpha} = \max_\alpha$ such that $S_\alpha \subseteq \mathcal{X}$.

In Hovd et al. (2014) and Munir et al. (2016), a controller based on (2) with polytopic controlled contractive sets $S = \{x_k | F x_k \leq 1\}$ were studied based on a piecewise linear function

$$V(x_k) = max\{F x_k\} \tag{3}$$

Using the function specified as in (3), the optimization (2) becomes a standard quadratic program, which may be solved parametrically with x_k and $V(x)$ as parameters. This is done by imposing a virtual parameter $\alpha_k = V(x_k)$ before solving the optimization in (2) at time k. The constraint in (2d) then simply becomes $F(Ax_k + Bu_k) \leq \gamma\alpha_k$. As the total number of constraints and the number of degrees of freedom are typically quite modest in (2) compared to a classical MPC problem utilizing a longer prediction horizon, the parametric solution is also of modest complexity. However, this approach suffers from the drawbacks described in the Introduction, and this paper therefore focuses on allowing more general types of function $V(x)$, to obtain a larger operating region with modest online computational complexity and memory requirement for the control.

In the developments below, two ellipsoidal controlled contractive sets will be important as terms of comparison:

- The set $\Omega = \{x \in \Re^{n_x} | x^T P^{-1} x \leq 1\}$, the largest controlled γ-contractive set that can be obtained using linear state feedback.
- The set $\Omega_{uc} = \{x \in \Re^{n_x} | x^T P_{uc}^{-1} x \leq 1\}$, the ellipsoidal set where γ-contractiveness is achieved with the linear state feedback $u_k = K_{uc} x_k$.

Constraints in both states and inputs are accounted for in the calculation of both Ω and Ω_{uc}. These sets can be calculated using well known techniques based on Linear Matrix Inequalities, see, *e.g.*, Blanchini and Miani (2008) or Ngyuyen (2012) for details.

While the set Ω_{uc} can be found for any given controller[1] K_{uc}, for the subsequent use in this paper it will be considered to be the unconstrained solution to (2), see the inverse optimality arguments in Nguyen, Olaru, Rodriguez-Ayerbe, and Hovd (2014) for the choice of weights Q and R. When ignoring the input and contractivity constraints, (2) yields the controller

$$u_k = \underbrace{-(R + B^T Q B)^{-1} B^T Q A}_{K_{uc}} x_k$$

For notational convenience in the following, we will define $P_1 = P^{-1}$ and $P_0 = P_{uc}^{-1}$.

[1] For subsequent developments to make sense, the controller K_{uc} should clearly be designed such that the unconstrained closed loop system is γ-contractive, *i.e.*, such that $max\ |eig(A + BK)| \leq \sqrt{\gamma}$.