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Evenly convex credal sets ☆

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ABSTRACT

An evenly convex credal set is a set of probability measures that is evenly convex; that is, a set that is an arbitrary intersection of open halfspaces. An evenly convex credal set can for instance encode preference judgments through strict and non-strict inequalities such as $\mathbb{P}(A) > 1/2$ and $\mathbb{P}(A) \leq 2/3$. This paper presents an axiomatization of evenly convex sets from preferences, where we introduce a new (and very weak) Archimedean condition. We examine the duality between preference orderings and credal sets; we also consider assessments of almost preference and natural extensions. We then discuss regular conditioning, a concept that is closely related to evenly convex sets.

1. Introduction

The goal of this paper is to show, first, that relatively simple axioms on preference orderings can be used to characterize *evenly convex* sets of probability measures; that is, sets that are arbitrary intersections of open halfspaces. Evenly convex sets allow assessments such as $\mathbb{P}(A) \geq 1/2$ and $1/4 < \mathbb{P}(B) \leq 3/4$: strict and non-strict inequalities can be expressed on probability values. Central to our results is a new (very weak) Archimedean condition. We then examine the definition of conditioning under such axioms, as well as concepts of almost preference and natural extension.

A preference ordering is a binary relation $\succ$ on *gambles*; a gamble is a function X that yields a real number $X(\omega)$ for each *state* ω, and $X \succ Y$ is understood as "X is preferred to Y". If a preference ordering satisfies a few conditions, to be discussed later, then the ordering can be represented by a single probability measure (that is, by an additive set-function that assigns a nonnegative number to each event, such that $\mathbb{P}(\Omega) = 1$). It is not always reasonable to assume that a precise probability value can be attached to every possible event: one might be willing to attach probability values only to a few events, or perhaps one might associate probability intervals with events, or even impose weaker constraints on probability values.

If a preference ordering is only a partial order, then, subject to a few additional conditions, it can be represented by a set of probability measures [16,28,33,35,36]. Typically such axiomatizations of sets of probability measures focus on *maximal closed convex* sets of probability measures. It seems that the only existing axiomatization that allows for open sets of probability measures sets has been given by Seidenfeld, Schervish, and Kadane [29], using a general setting where utilities are also derived, and a proof technique based on transfinite induction. Their representation result may require sets of state-dependent utilities to represent preferences; for this reason it may be a little difficult to grasp the geometric content of a preference profile. One wonders whether it is possible to capture assessments such as $\mathbb{P}(A) > 1/2$ with some intuitive construction.

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Section 3 presents our axiomatization for evenly convex sets of probability measures. We use the new Archimedean condition, and emphasize the use of separating hyperplanes as much as possible, hopefully producing results that can be appreciated with moderate effort. We study the connection of our Archimedean condition with other conditions in the literature, and we examine the duality between preference orderings and sets of probability measures. Section 4 looks at assessments of "almost preference", and studies the natural extension of sets of assessments. Finally, Section 5 discusses *regular conditioning*, a popular form of conditioning that is intimately related to evenly convex sets.

2. Preference orderings, sets of desirable gambles, and credal sets

In this section we present some basic concepts and results used throughout. Because some results here are in essence well-known, only very short proofs are given for them.

Consider a finite set Ω containing n states $\{\omega_1, \dots, \omega_n\}$. An *event* is a subset of Ω; a *gamble* is a function $X : \Omega \to \Re$. A gamble can be viewed as a n-dimensional vector. A probability measure over Ω is entirely specified by a n-dimensional vector with non-negative elements that add up to one. Given such a vector p that induces a probability measure $\mathbb{P}$, and a gamble X, the expected value of X, denoted by $\mathbb{E}_{\mathbb{P}}[X]$, is simply the inner product $X \cdot p$.

All sets we consider are subsets of $\Re^n$; throughout we assume the Euclidean topology. For a set $\mathcal{A}$, $\mathsf{cl}\mathcal{A}$ is the closure of $\mathcal{A}$ and $\mathsf{relint}\mathcal{A}$ is the relative interior of $\mathcal{A}$. A *cone* $\mathcal{A}$ is a set such that if $X \in \mathcal{A}$ then $\lambda X \in \mathcal{A}$ for $\lambda > 0$ (the origin may not be in $\mathcal{A}$). If $\mathcal{B}$ is a convex set, the *smallest convex cone* containing $\mathcal{B}$ is $\{\lambda X : \lambda > 0, X \in \mathcal{B}\}$ [27, Corollary 2.6.3]. An *exposed ray* of a convex cone is an exposed face that is a half-line emanating from the origin (recall that an exposed face is a face that is equal to the set of points achieving the maximum of some linear function).

Most results in this paper deal with the representation of preferences[1]:

Definition 1. A *preference ordering* $\succ$ is a strict partial order over pairs of gambles.

Absence of preference between X and Y is indicated by $X \not\sim Y$. If $X \succ 0$, X is *desirable*; if $X \not\sim 0$, X is *neutral*.

We always assume two additional properties:

Monotonicity: If $X(\omega) > Y(\omega)$ for all $\omega \in \Omega$, then $X \succ Y$;
Cancellation: For all $\alpha \in (0, 1]$, $X \succ Y$ iff $\alpha X + (1-\alpha)Z \succ \alpha Y + (1-\alpha)Z$.

The following representation obtains:

Proposition 2. *If a preference ordering $\succ$ satisfies monotonicity and cancellation, then there is a convex cone $\mathcal{D}$, not containing the origin but containing the interior of the positive orthant, such that $X \succ Y$ iff $X - Y \in \mathcal{D}$.*

The proof is short and instructive:

Proof. First, $X \succ Y$ iff $(X+Z)/2 \succ (Y+Z)/2$ iff $(X+Z)/4+0/2 \succ (Y+Z)/4+0/2$ iff $X+Z \succ Y+Z$ (applying cancellation). Hence $X \succ Y$ iff $X - Y \succ 0$. Also $Y \succ 0$ iff $0 \succ -Y$; if $X \succ 0$ and $Y \succ 0$ we have $X \succ 0 \succ -Y$ and by transitivity we obtain $X \succ -Y$, thus $X + Y \succ 0$. If $X \succ 0$, then $\lambda X \succ 0$ for any $\lambda \in (0, 1]$ by cancellation, and by finite induction we get $\lambda X \succ 0$ for any $\lambda > 0$. Hence $\succ$ can be represented by a cone that contains every positive gamble (by monotonicity) and does not contain the zero gamble (because $X \succ X$ is not allowed). □

As shown in this proof, we can capture a preference ordering by focusing on preferences with respect to the zero gamble, or, equivalently, by focusing on a convex cone of gambles. Cones that encode preference orderings have received attention for some time [16,28,33,35,36]. The literature on *sets of desirable gambles* [22,25,34] employs cones of gambles to model preferences, often assuming the following property of *admissibility*: if $X(\omega) \geq Y(\omega)$ for all ω and $X(\omega) > Y(\omega)$ for some ω, then $X \succ Y$. We do not assume admissibility here.[2]

In this paper we use the term *set of desirable gambles* to refer to a convex cone $\mathcal{D}$ that represents a preference ordering as in Proposition 2. This proposition allows one to freely switch between preference orderings and sets of desirable gambles.

One might think that *any* convex cone of gambles could be represented by a set $\mathcal{K}$ of probability measures as follows: $X \in \mathcal{D}$ iff $\mathbb{E}_{\mathbb{P}}[X] > 0$ for all $\mathbb{P} \in \mathcal{K}$. This is not possible. Consider the set of desirable gambles depicted in Fig. 1 (left).[3] All

[1] A *strict partial order* is a binary relation that is irreflexive and transitive; an *equivalence* is a binary relation that is reflexive, transitive, and symmetric (a binary relation $\diamond$ is *irreflexive* when $X \diamond X$ is false for every X; it is *transitive* when $X \diamond Y$ and $Y \diamond Z$ imply $X \diamond Z$; it is *symmetric* when $X \diamond Y$ implies $Y \diamond X$) [15, Section 2.3].

[2] Admissibility cannot be satisfied in general if preferences are to be encoded by expectation with respect to probability measures that may assign probability zero to events. That is, suppose we want to have that $X \succ Y$ iff $\mathbb{E}_{\mathbb{P}}[X] > \mathbb{E}_{\mathbb{P}}[Y]$; we may face X and Y such that $X(\omega) = Y(\omega)$ for all ω except that $X(\omega') > Y(\omega')$ for ω' with $\mathbb{P}(\omega') = 0$ (in this case $X \succ Y$ due to admissibility but $\mathbb{E}_{\mathbb{P}}[X] = \mathbb{E}_{\mathbb{P}}[Y]$).

[3] In all figures, the interior of sets of desirable gambles appear in pink, and their boundaries appear in red. Sets of probability measures appear in orange or purple; other sets appear in blue.