



Strain energy spectral density and information content of materials deformation

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ABSTRACT

A broader range of analytical tools can enhance understanding of the unusual mechanical properties of metamaterials and other advanced material systems. Here, we discuss a mechanics analogue of the Parseval's energy theorem that leads to a density of the strain energy in the reciprocal space. It reflects the ways for an elastic medium to translate static deformation patterns between two points in space. A normalized spectral density also provides an information entropy of deformation at those points. Both differential and discrete (numerical) entropies of the Shannon's type are discussed. Spectral entropy is a basic measure of information available in the material interior about surface loads, or a measure of disorder introduced into elastic medium by the deformation. An exact analytical entropy function is derived for an isotropic plane solid under Gauss-distributed and point loads. Approaches to numerical calculation of spectral entropy in computational solid mechanics are also discussed. Energy spectral density and spectral entropy of an elastic continuum is shown to translate, logically, in agreement with the Saint-Venant's principle. However, it also becomes clear that microstructured media may demonstrate anomalous pathways of evolution of the strain energy spectrum, enabling interesting transformation mechanics studies of engineered material systems.

1. Introduction

Recent advances in mechanical metamaterials create interesting opportunities for the control of strain energy distribution in deformable bodies. Mechanical metamaterials [1–10] are structural composites that manifest behaviors beyond the scope of traditional mechanics of materials, including negative elastic moduli [1–7] and basic symmetries breaking [8,9]. Identification of dangerous stress and strain profiles in autonomous materials systems, and pre-programmed processing and modification of surface loads in the materials interior are some examples of their potentially diverse applications. In a broader sense, control and analysis of spectral content of static deformation is an interesting subject that has been widely overlooked in materials mechanics. In this paper, we demonstrate a mechanics analogue of the Parseval's theorem from the field of information theory and signal processing. This theorem gives a *strain energy spectral density* (SESD), whose spatial evolution provides a fundamental insight on how mechanical stresses are transformed and modified in the material interior. The strain energy spectral density also enables a straightforward calculation of a spectral entropy of static deformation and its variance with distance to loads. Spectral entropy is a measure of smoothness of the energy spectrum and therefore can indicate proximity of defects in the material, or unusual and potentially hazardous load patterns. We also show that the spectral entropy has a

simple relation to the Shannon's information entropy [11,12], and therefore, it can serve as a measure of information contained in the material deformation about any available surface and volume forces.

Spectral energy density and the corresponding spectral entropy of deformation for a Gauss-type pressure load acting on surface of a plane solid are derived analytically as functions of distance to the load. A discussion is also provided on approaches to numerical calculation of the spectral entropy in general solid mechanics problems. The reader will clearly see what information is contained in mechanical deformation of a homogenous elastic continuum, and how it changes when the strain energy spectrum evolves in the material interior with distance to the surface load. Also, approaches presented here can be extended to analyze energy and entropic properties of various classes of advanced microstructured materials, and even mechanical metamaterials, including those featuring the non-reciprocity of mechanical deformation [9] and the Saint-Venant's effect reversal [8]. These material systems may demonstrate anomalous strain energy transformation capabilities and enable a wide range of interesting practical opportunities.

2. Parseval's theorem and strain energy spectral density

A mechanics version of the Parseval's theorem, also known as energy theorem of the Fourier transform, see references [13–15] and Eq. (A.1) in Appendix, can be written in a variety of ways. For example, we

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could consider a standard volumetric or spatial density of strain energy for an arbitrary plane stress problem, as a quadratic form of the strain components,

$$W(x, y) = \frac{1}{2} \boldsymbol{\varepsilon}^*(x, y) \mathbf{E} \boldsymbol{\varepsilon}(x, y) \quad (1)$$

where \mathbf{E} is a constitutive matrix (A.15), and $\boldsymbol{\varepsilon}^*$ is a conjugate transpose of the vector of strain components,

$$\boldsymbol{\varepsilon}(x, y) = \begin{Bmatrix} \varepsilon_x(x, y) \\ \varepsilon_y(x, y) \\ \gamma_{xy}(x, y) \end{Bmatrix} \quad (2)$$

and then integrate (1) over the y -coordinate,

$$\Pi(x) = \int_{-\infty}^{\infty} W(x, y) dy \quad (3)$$

This integral gives a distribution of the strain energy in the x -axis direction. We may represent the vector (2) via the inverse Fourier transform (A.3) of its own Fourier image (A.2),

$$\tilde{\boldsymbol{\varepsilon}}(x, q) = \int_{-\infty}^{\infty} \boldsymbol{\varepsilon}(x, y) e^{-iqy} dy \quad (4)$$

and rewrite (3) as the following:

$$\Pi(x) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \tilde{\boldsymbol{\varepsilon}}^*(x, q') e^{-iq'y} dq' \right) \mathbf{E} \left(\int_{-\infty}^{\infty} \tilde{\boldsymbol{\varepsilon}}(x, q) e^{iqy} dq \right) dy \quad (5)$$

Next, rearrange this expression by changing the integration order,

$$\begin{aligned} \Pi(x) &= \frac{1}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-iq'y} e^{iqy} dy \right) \tilde{\boldsymbol{\varepsilon}}^*(x, q') \mathbf{E} \tilde{\boldsymbol{\varepsilon}}(x, q) dq dq' \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(q - q') \tilde{\boldsymbol{\varepsilon}}^*(x, q') \mathbf{E} \tilde{\boldsymbol{\varepsilon}}(x, q) dq dq' \end{aligned} \quad (6)$$

Here, δ is the Dirac's delta function (Eqs. A.8) and (A.9) which removes the integral over q' and gives finally

$$\Pi(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \tilde{\boldsymbol{\varepsilon}}^*(x, q) \mathbf{E} \tilde{\boldsymbol{\varepsilon}}(x, q) dq \quad (7)$$

A comparison of (3) and (7) proves the **strain energy spectral theorem**: *An integral of strain energy volumetric density W is equal to a Fourier integral of strain energy spectral density \tilde{W} . Equivalently, an integral over a quadratic form of strain components is equal to an integral over the quadratic form of Fourier transforms of these strain components:*

$$\int_{-\infty}^{\infty} W(x, y) dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{W}(x, q) dq \quad (8)$$

$$W(x, y) = \frac{1}{2} \boldsymbol{\varepsilon}^*(x, y) \mathbf{E} \boldsymbol{\varepsilon}(x, y) \quad (9)$$

$$\tilde{W}(x, q) = \frac{1}{2} \tilde{\boldsymbol{\varepsilon}}^*(x, q) \mathbf{E} \tilde{\boldsymbol{\varepsilon}}(x, q) \quad (10)$$

The strain energy spectral density (SESD) function, $\tilde{W}(x, q)$, is a very interesting characteristic of a state of deformation. Dependence of this function on q shows a spectral density (spectral distribution) of the strain energy, and it tells us how much strain energy is contained in the wavenumber interval from q to $q + dq$. Furthermore, its dependence on x determines how this spectral density transforms from one point in space to another.

Considering a general three-dimensional energy density $W(x, y, z)$ and applying single or multiple integrals similar to (3), we may also get functions of the type $\tilde{W}(x, y, q)$, as well as $\tilde{W}(x, q_1, q_2)$ and a full transform $\tilde{W}(q_1, q_2, q_3)$ under a double or triple Fourier integral in (8). Examples are shown in Appendix (Eqs. (A.10–A.12)). The function $\tilde{W}(x, q_1, q_2)$ will show a two-dimensional spectral distribution of the strain energy contained in the layer $(x, x + dx)$ of the material. However, in the analysis of material responses to surface loads, strain energy spectral density of the type (10) or (A.11) depending on a *single* coordinate, as a distance to the loads, will be most interesting in the practice of transformation mechanics studies.

For the discussion to follow, we also introduce the normalized spatial (w) and spectral (\tilde{w}) densities,

$$w(x, y) = \frac{W(x, y)}{\Pi(x)} \quad (11)$$

$$\tilde{w}(x, q) = \frac{1}{2\pi} \frac{\tilde{W}(x, q)}{\Pi(x)} \quad (12)$$

such that

$$\int_{-\infty}^{\infty} w(x, y) dy = \int_{-\infty}^{\infty} \tilde{w}(x, q) dq = 1 \quad (13)$$

Transformation or evolution of the strain energy spectral density in space (with the coordinate x) will be specific to a given material, and a recent study [8] shows that microstructured materials may be highly selective to certain spectral components of their deformation patterns. In particular, some Fourier modes may dissipate unexpectedly fast in the material volume or even get localized on the surface. The present formalism can help to understand these interesting behaviors, and in Section 6, we initiate the discussion by considering an isotropic continuum as a reference case.

3. Differential spectral entropy

Knowledge of a normalized spectral density of strain energy (12) enables us to determine the corresponding spectral entropy. **Strain energy spectral entropy (SESE)**, as a function of the coordinate x , can be written using the Shannon's definition of differential entropy [11,12],

$$S(x) = - \int_{-\infty}^{\infty} \tilde{w}(x, q) \ln \tilde{w}(x, q) dq \quad (14)$$

The functional S is a measure of *complexity of the strain field*, or a measure of *disorder introduced to the elastic continuum by mechanical forces*.

The strain energy spectral entropy (14) will generally be higher near stress concentrators, localized surface loads or a physical inhomogeneity in the form of voids, cracks and secondary phase inclusions. Indeed, stress concentrators give volumetric strain energy density functions W highly localized in space, whose reciprocal space counterpart \tilde{W} , on the contrary, will be a smooth function of the wavenumber q . Although \tilde{W} is not a Fourier transform of W , this example is insightful: Fourier transform of a Gaussian function is also a Gaussian function, but with an inverse width,

$$\tilde{w}_G(q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{a^2 y^2}{2}} e^{-iqy} dy = \frac{1}{a\sqrt{2\pi}} e^{-\frac{q^2}{2a^2}} \quad (15)$$

In simple terms, a narrow function in the coordinate space generally corresponds to a wide function in the reciprocal space, and vice versa. We further note that differential entropy of the normal distribution (15) is a logarithmic function of its width a ,

$$S_G = - \int_{-\infty}^{\infty} \tilde{w}_G(q) \ln \tilde{w}_G(q) dq = \ln \left(a\sqrt{2\pi e} \right) \quad (16)$$

Therefore, theoretical range of the differential entropy integral (14) is from $-\infty$ to ∞ , when the width a of the normal distribution varies from 0 to ∞ . Moreover, if the material coordinates are not dimensionless originally, the function $\tilde{w}(x, q)$ will have a physical dimensionality of length. Then, a dimensionless coordinate y/Λ should be introduced leading to a scaling factor $1/\Lambda$ for $\tilde{w}(x, q)$, and because of the negative logarithm in (14) and property (13), this will produce a shift of the entire entropy function (14) by a constant proportional to $\ln \Lambda$. Thus, only a relative change of the differential entropy (14) between two points in a coordinate space can have a physical meaning, while its absolute value will generally depend on a physical scaling of that coordinate space.

4. Numerical entropy calculation

Fortunately, the unlimited range of the differential entropy of mechanical deformation (14), varying from $-\infty$ to ∞ , and uncertainty of

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