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Geodesic orbit spheres and constant curvature in Finsler geometry ☆

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ABSTRACT

In this paper, we generalize the classification of geodesic orbit spheres from Riemannian geometry to Finsler geometry. Then we further prove if a geodesic orbit Finsler sphere has constant flag curvature, it must be Randers. It provides an alternative proof for the classification of invariant Finsler metrics with $K \equiv 1$ on homogeneous spheres other than $Sp(n)/Sp(n-1)$.

1. Introduction

A Riemannian homogeneous manifold is called a *geodesic orbit space*, if any geodesic is the orbit of a one-parameter subgroup of isometries. This notion was introduced by O. Kowalski and L. Vanhecke in 1991 [24], as a generalization of naturally reductive homogeneity. Since then, there have been many research works on this subject. See [1][2][3][8][14][17][18] for example.

Meanwhile, geodesic orbit metrics have also been studied in Finsler geometry. In [33], the notion of geodesic orbit Finsler space was defined, and in [29], the interaction between geodesic orbit property and negative curvature property was explored.

The first purpose of the paper is to generalize Yu.G. Nikonorov's classification of geodesic orbit metrics on spheres [25] to Finsler geometry, and prove the following theorem.

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Theorem 1.1. *A homogeneous Finsler metric F on a sphere $M = S^n$ with $n > 1$ is a geodesic orbit metric iff the connected isometry group $I_o(M, F)$ is not isomorphic to $Sp(k)$ for any $k \geq 1$.*

By this theorem, we can easily list all the geodesics orbit metrics on spheres:

(1) Riemannian metrics of constant curvature.
(2) All homogeneous Finsler metrics on $S^{4n-1} = Sp(n)Sp(1)/Sp(n-1)Sp(1)$ with $n > 1$ and $S^{15} = Spin(9)/Spin(7)$. They are all of (α_1, α_2)-type, in which some special ones are Riemannian.
(3) All homogeneous Finsler metrics on $SU(n)/SU(n-1)$ with $n > 2$ and $U(n)/U(n-1)$ with $n > 1$. They are all of (α, β)-type, in which some special ones are Riemannian.
(4) All homogeneous Finsler metrics on $Sp(n)U(1)/Sp(n-1)U(1)$. They are all of $(\alpha_1, \alpha_2, \beta)$-type, in which some special ones are of (α, β)-type, (α_1, α_2)-type, or Riemannian.

See Section 2 for the notions of these metrics. When the metrics are Riemannian, the above list re-verifies Table 1 in [25]. For each case of (2)–(4), the space of geodesic orbit metrics has an infinite dimension. In an independent work [34], S. Zhang and S. Deng classified geodesic orbit Randers spheres with a more algebraic method, and described their geodesic vectors.

The second purpose of this paper is to apply Theorem 1.1 to homogeneous Finsler spheres of constant flag curvature $K \equiv 1$, and explore the interaction between geodesic orbit property and constant positive curvature property. We will prove the following theorem.

Theorem 1.2. *A homogeneous Finsler sphere $(M, F) = (S^n, F)$ with $n > 1$ and $K \equiv 1$ is a geodesic orbit space iff it is Randers.*

All Randers spheres $(M, F) = (S^n, F)$ with $n > 1$ and $K \equiv 1$ are classified by D. Bao, Z. Shen and C. Robles [9], i.e. the metric F must be defined by the navigation datum (h, W), in which h is the Riemannian metric for the unit sphere, and W is a Killing vector field with $h(W, W) < 1$ everywhere. The only new ingredient is that F is homogeneous iff W has a constant h-length. So we have the following corollary of Theorem 1.1 and Theorem 1.2,

Corollary 1.3. *Any invariant Finsler metric F on a homogeneous sphere M with $\dim M > 1$, $K \equiv 1$ and $I_o(M, F) \neq Sp(k)$ for all $k \in \mathbb{N}$, i.e. $M \neq Sp(n)/Sp(n-1)$ for all $n > 1$ or $Sp(1)/Sp(0) = SU(2)/\{e\}$ in the list (4.1) of homogeneous spheres, then F is a Randers metric defined by the navigation datum (h, W) in which h is the Riemannian metric for the unit sphere and W is a Killing vector field of constant length on (M, h).*

Corollary 1.3 has classified all the invariant Finsler metrics with constant flag curvature on a homogeneous sphere, except for the most difficult case $M = Sp(n)/Sp(n-1)$. It implies a negative answer to the question if a homogeneous Finsler metric of constant flag curvature can be “exotic”. Global homogeneity for the metric is a critical condition, because in the non-homogeneous situation, we know the examples discovered by R.L. Bryant [4][5][6], and there may exist many more.

The above theorems and corollary can also be applied to study a homogeneous Finsler sphere (M, F) with $K \equiv 1$ and finite orbits of prime closed geodesics [28]. By Theorem 1.2 in this paper and that in [28], we may find totally geodesic sub-manifolds of M which are Randers spheres.

By private communication, the author noticed that L. Huang had discovered Corollary 1.3 in 2015, and found a computational proof based on his homogeneous flag curvature formula [19][20]. The method in this paper is more geometrical, and has not used any calculation concerning L. Huang formula. A Killing navigation process has been applied to reduce our discussion to the case that (M, F) is a geodesic orbit

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