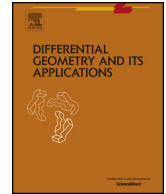




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## Differential Geometry and its Applications

[www.elsevier.com/locate/difgeo](http://www.elsevier.com/locate/difgeo)Geodesic orbit spheres and constant curvature in Finsler geometry <sup>☆</sup>

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## ABSTRACT

In this paper, we generalize the classification of geodesic orbit spheres from Riemannian geometry to Finsler geometry. Then we further prove if a geodesic orbit Finsler sphere has constant flag curvature, it must be Randers. It provides an alternative proof for the classification of invariant Finsler metrics with  $K \equiv 1$  on homogeneous spheres other than  $Sp(n)/Sp(n-1)$ .

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## 1. Introduction

A Riemannian homogeneous manifold is called a *geodesic orbit space*, if any geodesic is the orbit of a one-parameter subgroup of isometries. This notion was introduced by O. Kowalski and L. Vanhecke in 1991 [24], as a generalization of naturally reductive homogeneity. Since then, there have been many research works on this subject. See [1][2][3][8][14][17][18] for example.

Meanwhile, geodesic orbit metrics have also been studied in Finsler geometry. In [33], the notion of geodesic orbit Finsler space was defined, and in [29], the interaction between geodesic orbit property and negative curvature property was explored.

The first purpose of the paper is to generalize Yu.G. Nikonorov's classification of geodesic orbit metrics on spheres [25] to Finsler geometry, and prove the following theorem.

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**Theorem 1.1.** *A homogeneous Finsler metric  $F$  on a sphere  $M = S^n$  with  $n > 1$  is a geodesic orbit metric iff the connected isometry group  $I_o(M, F)$  is not isomorphic to  $Sp(k)$  for any  $k \geq 1$ .*

By this theorem, we can easily list all the geodesics orbit metrics on spheres:

- (1) Riemannian metrics of constant curvature.
- (2) All homogeneous Finsler metrics on  $S^{4n-1} = Sp(n)Sp(1)/Sp(n-1)Sp(1)$  with  $n > 1$  and  $S^{15} = Spin(9)/Spin(7)$ . They are all of  $(\alpha_1, \alpha_2)$ -type, in which some special ones are Riemannian.
- (3) All homogeneous Finsler metrics on  $SU(n)/SU(n-1)$  with  $n > 2$  and  $U(n)/U(n-1)$  with  $n > 1$ . They are all of  $(\alpha, \beta)$ -type, in which some special ones are Riemannian.
- (4) All homogeneous Finsler metrics on  $Sp(n)U(1)/Sp(n-1)U(1)$ . They are all of  $(\alpha_1, \alpha_2, \beta)$ -type, in which some special ones are of  $(\alpha, \beta)$ -type,  $(\alpha_1, \alpha_2)$ -type, or Riemannian.

See Section 2 for the notions of these metrics. When the metrics are Riemannian, the above list re-verifies Table 1 in [25]. For each case of (2)–(4), the space of geodesic orbit metrics has an infinite dimension. In an independent work [34], S. Zhang and S. Deng classified geodesic orbit Randers spheres with a more algebraic method, and described their geodesic vectors.

The second purpose of this paper is to apply Theorem 1.1 to homogeneous Finsler spheres of constant flag curvature  $K \equiv 1$ , and explore the interaction between geodesic orbit property and constant positive curvature property. We will prove the following theorem.

**Theorem 1.2.** *A homogeneous Finsler sphere  $(M, F) = (S^n, F)$  with  $n > 1$  and  $K \equiv 1$  is a geodesic orbit space iff it is Randers.*

All Randers spheres  $(M, F) = (S^n, F)$  with  $n > 1$  and  $K \equiv 1$  are classified by D. Bao, Z. Shen and C. Robles [9], i.e. the metric  $F$  must be defined by the navigation datum  $(h, W)$ , in which  $h$  is the Riemannian metric for the unit sphere, and  $W$  is a Killing vector field with  $h(W, W) < 1$  everywhere. The only new ingredient is that  $F$  is homogeneous iff  $W$  has a constant  $h$ -length. So we have the following corollary of Theorem 1.1 and Theorem 1.2,

**Corollary 1.3.** *Any invariant Finsler metric  $F$  on a homogeneous sphere  $M$  with  $\dim M > 1$ ,  $K \equiv 1$  and  $I_o(M, F) \neq Sp(k)$  for all  $k \in \mathbb{N}$ , i.e.  $M \neq Sp(n)/Sp(n-1)$  for all  $n > 1$  or  $Sp(1)/Sp(0) = SU(2)/\{e\}$  in the list (4.1) of homogeneous spheres, then  $F$  is a Randers metric defined by the navigation datum  $(h, W)$  in which  $h$  is the Riemannian metric for the unit sphere and  $W$  is a Killing vector field of constant length on  $(M, h)$ .*

Corollary 1.3 has classified all the invariant Finsler metrics with constant flag curvature on a homogeneous sphere, except for the most difficult case  $M = Sp(n)/Sp(n-1)$ . It implies a negative answer to the question if a homogeneous Finsler metric of constant flag curvature can be “exotic”. Global homogeneity for the metric is a critical condition, because in the non-homogeneous situation, we know the examples discovered by R.L. Bryant [4][5][6], and there may exist many more.

The above theorems and corollary can also be applied to study a homogeneous Finsler sphere  $(M, F)$  with  $K \equiv 1$  and finite orbits of prime closed geodesics [28]. By Theorem 1.2 in this paper and that in [28], we may find totally geodesic sub-manifolds of  $M$  which are Randers spheres.

By private communication, the author noticed that L. Huang had discovered Corollary 1.3 in 2015, and found a computational proof based on his homogeneous flag curvature formula [19][20]. The method in this paper is more geometrical, and has not used any calculation concerning L. Huang formula. A Killing navigation process has been applied to reduce our discussion to the case that  $(M, F)$  is a geodesic orbit

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