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Discretization error in the random finite element method for spatially variable undrained shear strength

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ABSTRACT

This study decomposes the discretization error in the random finite element method into finite element and random field discretization errors. Two numerical problems are considered: a soil column and a retaining wall with spatially variable undrained shear strength. It is observed that these two errors tend to accumulate if the spatial averaging (SA) method is adopted for discretization, whereas the two errors may not accumulate if the midpoint (MP) method is adopted. Therefore, MP may outperform SA. Suggestions for mesh sizes are provided, but these suggestions may be restricted to these two numerical problems.

1. Introduction

The random finite element method (RFEM) [1–5] is increasingly used in geotechnical engineering. The interest in the RFEM has stemmed from the fact that the spatial variability of soil properties can play a key role in the behavior of geotechnical structures. The RFEM explicitly accounts for spatial variability by combining random fields [6], finite element method (FEM), and Monte Carlo simulation. In the context of spatial variability in shear strength, the effect of spatial variability goes beyond the so-called averaging effect [6], which reduces the variance of the response. The spatial variability in shear strength also leads to nonclassical failure mechanisms, because the failure path tends to seek out weak zones (e.g., [7,8]). The effect of seeking out weak zones shifts the mean response to the unconservative side (e.g., [9–12]).

The RFEM must meet a number of challenges for its implementation. A major challenge is the discretization error. The discretization error is the discrepancy between the true solution for the continuous mathematical model and the approximate RFEM solution. In the RFEM, this error may come from three possible sources: (a) random field (RF) discretization error, (b) RF truncation error, and (c) finite element (FE) discretization error:

1. The RF discretization error and RF truncation error: The RF discretization error and RF truncation error are purely associated with RF simulations. Some RF simulation methods induce a discretization error (e.g., the local averaging subdivision (LAS) [13]), while some others induce a truncation error (e.g., the Karhunen–Loève (KL) expansion [14]). Note that RFEM methods that use the KL expansion not only have the RF truncation error but also the RF discretization error. This is because when the random field simulated by the KL expansion is fed into FEM, the FE mesh cannot accommodate a continuous random field. The random field needs to be discretized even if the KL expansion is adopted.
2. The FE discretization error: The FE discretization error is purely associated with FEM. The FE discretization error is due to the mesh discretization of stress/strain and exists even in deterministic FEM.

This study focuses on the RF discretization error and FE discretization error. The RF truncation error is not the focus, because this error is minimized during the RF simulation process in this study.

In the area of the stochastic finite element method (SFEM) in the structural reliability community, the issue of discretization errors has been extensively investigated. Note that the RFEM is in fact a Monte Carlo-based SFEM, and both methods face the same discretization challenge. Cherng and Wen [15] stated that the required element size to control discretization errors depends on the structural type, response of interest, discretization method, and correlation structure of the random field (both to be explained in a later section). A dimensionless ratio of the element size to the scale of fluctuation (SOF) is often reported in the literature, where the SOF quantifies the distance over which property values are significantly correlated. For a beam or a plate, the ratio should not exceed one-eighth to one-half (e.g., [16–19]). Since maintaining this ratio for small SOFs may be computationally burdensome, Liu [20] obtained an equation to estimate the solution of a fine mesh using the solution of a coarse mesh. The work was then extended in

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[21] to propose an adaptive random field mesh refinement. Deodatis and Shinozuka [22] eliminated the need for the RF discretization by deriving an exact expression for the finite element stiffness matrix in terms of weighted integrals of the random field. It was, however, argued that the method still employs a hidden discretization, and moreover, the work seems limited to elastic structures [23,24].

In contrast to the structural reliability, the issue has not received much attention in the geotechnical community, although soils exhibit far more spatial variability than structural materials do. A few studies have addressed the issue of discretization errors. Hicks and Onisiphorou [25] mapped random fields onto the integration points (Gauss points), instead of the elements themselves, to alleviate this issue. Spencer [26] suggested that (element size)/SOF ≈ ¼ is the practical maximum to keep the error acceptable. To the authors' knowledge, only two studies in the geotechnical community have been fully devoted to the issue of discretization errors for spatially variable undrained shear strength: Ching and Phoon [27] and Huang and Griffiths [14]. They both considered a similar problem (soil column subjected to axial compressive loading). However, their recommended ratios for (element size)/SOF are very different. In fact, Ching and Phoon's [27] recommendation is almost ten times smaller than Huang and Griffiths' [14]. Apart from this difference, what is less understood in both studies, and in past studies, is how the two components of the discretization error (the FE and RF discretization errors) interact with each other and affect the solution. Insights from this can help to reduce the discretization error in a more effective way. The present paper therefore aims to address the following questions:

1. Do the two components of the discretization error accumulate or compensate?
2. Which component is larger?
3. Do the answers to #1 and #2 depend on the RF discretization method such as the spatial averaging method and midpoint method?

These questions are investigated using two numerical geotechnical problems: the compressive strength of an undrained soil column and the active lateral force on an undrained retaining wall. Although both problems yield qualitatively similar conclusions, the conclusions may be restricted to these two numerical problems. More research is required to obtain generic conclusions. The present paper also examines the recommendations proposed by Ching and Phoon [27] and Huang and Griffiths [14] for these two numerical problems. In the end, suggestions are provided for the allowable mesh size to achieve a certain error tolerance for these two problems. These suggestions may be restricted to these two numerical problems. The suggestions may no longer be valid if, for example, a different autocorrelation model or a different response quantity of interest is considered.

2. Random field model for soil spatial variability

Vanmarcke [6] proposed that the spatial variability of soil can be modeled as a random field. In most applications, the random field is assumed to be second-order stationary. This assumption allows characterizing the random field model based on limited site investigation data. The characterization of a second-order stationary random field requires three parameters: (1) the mean, (2) variance, and (3) autocorrelation function. The first two are constant everywhere. The third, which defines the correlation between two points, is a function of their separation distance rather than their absolute positions. One of the common autocorrelation functions used in the geotechnical engineering literature is the single exponential model [6,28]. In two dimensions, it defines the correlation between two points with a separation distance of Δx and Δz as follows:

$$\rho(\Delta x, \Delta z) = \exp\left(-2\frac{|\Delta x|}{\delta_x} - 2\frac{|\Delta z|}{\delta_z}\right) \tag{1}$$

where δ_x and δ_z are, respectively, the SOFs in the x and z directions (x and z are the horizontal and vertical coordinates). Some of the other common autocorrelation functions are the squared exponential, cosine exponential, second order Markov, and binary noise models [28–30]. As mentioned by Cherng and Wen [15], the type of an autocorrelation function may influence the required element size. For example, Ching and Phoon [27] showed that the required element size for the single exponential model is considerably smaller than that for the squared exponential model. The present study does not address the effect of the autocorrelation function type and only considers the single exponential model with $\delta_x = \delta_z = \delta$. Note that Eq. (1) is a "separable" single exponential model, which is known to generate streaks along the horizontal and vertical directions. A "radial" single exponential model [26] is free from this issue, but it is not presented in the present study. Most of the qualitative conclusions in the present paper hold true for the radial single exponential model. However, the required element size for the radial form may be different than that for the separable form.

The Fourier series method (FSM) [31,32] is employed to simulate a two-dimensional stationary lognormal random field W(x,z) over a domain of size $L_x \times L_z$ with a point mean = μ and point variance = σ^2 (coefficient of variation = $V = \sigma/\mu$). This stationary lognormal random field simulation is achieved by taking the exponential of the underlying stationary normal random field with a mean = $\lambda = \ln[\mu/(1 + V^2)^{0.5}]$ and variance = $\xi^2 = \ln(1 + V^2)$:

$$W(x, z) = \exp\left(\lambda + \mathrm{Re}\left[\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(a_{mn} + ib_{mn})\exp\left(\frac{i2m\pi x}{L_x} + \frac{i2n\pi z}{L_z}\right)\right]\right) \tag{2}$$

where Re[.] denotes the real part of the enclosed complex number, and a_{mn} and b_{mn} are independent zero-mean normal random variables with variance σ^2_{mn}. For the separable single exponential model, σ^2_{mn} is given by [31]:

$$\sigma^2_{mn} = \frac{\xi^2}{q_x q_z}\left[\frac{1-\exp(-q_x)(-1)^m}{1 + m^2\pi^2/q_x^2}\right] \times \left[\frac{1-\exp(-q_z)(-1)^n}{1 + n^2\pi^2/q_z^2}\right] \tag{3}$$

where $q_x = L_x/\delta_x$ and $q_z = L_z/\delta_z$. Since the infinite sum in Eq. (2) must be truncated to a finite sum, Jha and Ching [31] suggested that it is sufficient to sum up to an $|m|$ value (or an $|n|$ value) corresponding to:

$$\frac{1}{q_x}\left[\frac{1-\exp(-q_x)(-1)^m}{1 + m^2\pi^2/q_x^2}\right] = 10^{-5} \tag{4}$$

without a noticeable RF truncation error. Therefore, it is expected that the impact of the RF truncation error is minimized, so the effect of the RF discretization error can be isolated in the present study.

In the midpoint (MP) discretization method, the field value for an element is represented by W(x,z) which is simulated at the centroid of the element. In the spatial averaging (SA) discretization method, the field value for an element is represented by the spatial average of the field over the element, W_{ave}. For the MP method, one can readily implement it in the FSM using Eq. (2). For the SA method, Jha and Ching [31] obtained an analytical expression in the FSM to directly simulate the spatial average of the normal random field over a rectangular element (see Eq. (16) in [31]), and the exponential of this average is taken. While both MP and SA methods preserve the mean of a normal field, the SA method reduces the mean of a lognormal field. In this case, a correction factor [2] $(1 + V^2)^{0.5\times(1-\Gamma 2)}$ is multiplied to W_{ave} to ensure the mean of the lognormal field remains unchanged, where Γ^2 is the variance reduction factor [6] for the spatial averaging effect within an element. For the single exponential model in Eq. (1), Γ^2 for an element with side lengths $\Delta x_e \times \Delta z_e$ is: