Mathematical and Computer Modelling 56 (2012) 49–55

Contents lists available at SciVerse ScienceDirect

Mathematical and Computer Modelling

journal homepage: www.elsevier.com/locate/mcm

A note on "A new approach for weight derivation using data envelopment analysis in the analytic hierarchy process"

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ARTICLE INFO

Article history:
Received 12 April 2011
Received in revised form 1 December 2011
Accepted 1 December 2011

Keywords:
DEAHP
Data envelopment analysis
Analytic hierarchy process
Weight derivation

ABSTRACT

DEAHP as a weight derivation procedure for analytic hierarchy process (AHP) has been found suffering from some significant drawbacks. Recently, Mirhedayatian and Saen (2011) [5] proposed a new procedure entitled Revised DEAHP for AHP weight derivation [S.M. Mirhedayatian, R.F. Saen, A new approach for weight derivation using data envelopment analysis in the analytic hierarchy process, Journal of the Operational Research Society 62 (2011) 1585–1595]. This paper provides a detailed note to reveal that (1) the Revised DEAHP cannot derive true weights from perfectly consistent pairwise comparison matrices, (2) it may produce irrational weights for inconsistent pairwise comparison matrices, (3) it still suffers from rank reversal problem when an efficient decision criterion or alternative is added or removed, (4) the use of the super-efficiency model in data envelopment analysis (DEA) for AHP weight derivation is redundant and meaningless when there exist multiple decision criteria or alternatives that are efficient in a pairwise comparison matrix, and (5) it may produce a completely reversed ranking that is totally opposite to the rank obtained by the eigenvector method in the case of hierarchical structures, leading to a wrong decision being made.

1. Introduction

How to derive priorities from pairwise comparison matrices has been being an important research topic in the analytic hierarchy process (AHP) and has been extensively investigated. Quite a number of approaches have been suggested and DEAHP is one of them, which was proposed by Ramanathan [1]. Such a method, however, has been found suffering from some significant drawbacks such as producing irrational weights for inconsistent pairwise comparison matrices. Detailed analyses and theoretical improvements can be found in [2–4].

Recently, Mirhedayatian and Saen [5] also analyzed the drawbacks of DEAHP that had been analyzed by Wang and Chin [2] and Wang et al. [3,4] and proposed a new procedure which they called Revised DEAHP for AHP weight derivation. Instead of the use of the CCR model [6] for AHP weight derivation, the Revised DEAHP applies the super-efficiency model [7] in data envelopment analysis (DEA) to improve the discriminating power of the DEAHP. In this paper, we provide a detailed note to illustrate with numerical examples the significant drawbacks that the Revised DEAHP suffers from.

The remainder of the paper is organized as follows. Section 2 briefly reviews the Revised DEAHP procedure. Section 3 examines the drawbacks that the Revised DEAHP suffers from. Section 4 concludes the paper with a brief summary.

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doi:10.1016/j.mcm.2011.12.001

2. Revised DEAHP

Let

$$A = (a_{ij})_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{1}$$

be a pairwise comparison matrix with $a_{ii} = 1$ and $a_{ji} = 1/a_{ij} > 0$ for $j \neq i$ and $W = (w_1, \dots, w_n)^T$ be a weight vector of the pairwise comparison matrix. To find the local weights of $A = (a_{ij})_{n\times n}$, the Revised DEAHP model first determines a lower bound value ε^* for multiplier variables v_j $(j = 1, \dots, n)$ by solving the following linear programming (LP) model, which is a variant of the CCR model in DEA:

$$\varepsilon^* = \text{Maximize } \varepsilon \tag{2}$$

$$\text{Subject to} \begin{cases} u_1 = 1, \\ \sum_{j=1}^{n} a_{ij}v_j - u_1 \le 0, & i = 1, \dots, n, \\ v_j \ge \varepsilon, & j = 1, \dots, n, \end{cases}$$

and then applies model (3) for weight derivation:

$$w_0^* = \text{Maximize } \sum_{j=1}^{n} a_{0j}v_j \tag{3}$$

$$\text{Subject to} \begin{cases} u_1 = 1, \\ \sum_{j=1}^{n} a_{ij}v_j - u_1 \le 0, & i = 1, \dots, n;\ i \neq 0, \\ v_j \ge \varepsilon^*, & j = 1, \dots, n, \end{cases}$$

where $i = 0$ represents the criterion or alternative under evaluation, and w_0 is its weight. Different from the DEAHP, model (3) excludes $w_0 - u_1 \le 1$ from its constraints and is a super-efficiency DEA model [7]. Model (3) is solved n times, each time for one different criterion or alternative. As a result, $W^* = (w_1^*, \dots, w_n^*)^T$ forms the optimal weight vector of the pairwise comparison matrix $A = (a_{ij})_{n\times n}$.

3. Comments on the revised DEAHP

In this section, we provide some detailed comments on the Revised DEAHP procedure to illustrate its significant drawbacks, from which it can be seen clearly that the use of the Revised DEAHP for AHP weight derivation is inappropriate.

Comment 1. *Model (2) produces equal weights for all the multipliers and the maximum value ε^* can be directly determined by $\varepsilon^* = \min_i\{1/\sum_{j=1}^{n} a_{ij}\}$ without the need to solve any programming.*

Proof. From $v_j \ge \varepsilon$ for $j = 1, \dots, n$, it can be derived that $\varepsilon \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij}\varepsilon \le \sum_{j=1}^{n} a_{ij}v_j \le u_1 = 1$. So, there exists $\varepsilon \le 1/\sum_{j=1}^{n} a_{ij}$ for all $i = 1, \dots, n$. That is, $\varepsilon \le \min_i\{1/\sum_{j=1}^{n} a_{ij}\}$. Quite obviously, the maximum value of ε is $\varepsilon^* = \min_i\{1/\sum_{j=1}^{n} a_{ij}\}$, which happens at $v_1^* = \cdots = v_n^* = \varepsilon^* = \min_i\{1/\sum_{j=1}^{n} a_{ij}\}$. □

Comment 2. *The Revised DEAHP cannot derive the true weights from perfectly consistent pairwise comparison matrices.*

Proof. Suppose that $A = (a_{ij})_{n\times n}$ is a perfectly consistent pairwise comparison matrix. Then, there must exist a set of true weights $\hat{w}_1^*, \dots, \hat{w}_n^*$ to satisfy $\hat{w}_i^* > 0$ for $i = 1, \dots, n$ and $\sum_{i=1}^{n} \hat{w}_i^* = 1$ such that $a_{ij} = \hat{w}_i^*/\hat{w}_j^*$ for $i, j = 1, \dots, n$. From the constraints of model (3), it can be derived that $\sum_{j=1}^{n} a_{ij}v_j = \sum_{j=1}^{n}(\hat{w}_i^*/\hat{w}_j^*)v_j \le u_1 = 1$, i.e. $\sum_{j=1}^{n}(v_j/\hat{w}_j^*) \le 1/\hat{w}_i^*$, which holds for $i = 1, \dots, n$, but $i \neq 0$, where $i = 0$ represents the criterion or alternative under evaluation. It is evident that $\sum_{j=1}^{n}(v_j/\hat{w}_j^*) \le \min_{i\neq 0}\{1/\hat{w}_i^*\} = 1/\max_{i\neq 0}\{\hat{w}_i^*\}$. Since the objective function of model (3) is for maximization, there is at least one constraint that is binding among the $(n-1)$ inequality constraints from $i = 1, \dots, n$, but $i \neq 0$. It can therefore be concluded that $\sum_{j=1}^{n}(v_j^*/\hat{w}_j^*) \equiv 1/\max_{i\neq 0}\{\hat{w}_i^*\}$ at optimality. Accordingly, the optimal objective function value of model (3) can be computed as $w_0^* = \sum_{j=1}^{n} a_{0j}v_j^* = \sum_{j=1}^{n}(\hat{w}_0^*/\hat{w}_j^*)v_j^* = \hat{w}_0^* \sum_{j=1}^{n}(v_j/\hat{w}_j^*) = \hat{w}_0^*/\max_{i\neq 0}\{\hat{w}_i^*\}$. Denote by $\hat{w}_{i_1}^*$ and $\hat{w}_{i_2}^*$ the biggest and the second biggest of the n true weights $\{\hat{w}_1^*, \dots, \hat{w}_n^*\}$. Then, we get

$$w_i^* = \frac{\hat{w}_i^*}{\max_{k\neq i}\{\hat{w}_k^*\}} = \begin{cases} \hat{w}_i^*/\hat{w}_{i_1}^*, & i = 1, \dots, n;\ i \neq i_1, \\ \hat{w}_i^*/\hat{w}_{i_2}^*, & i = i_1. \end{cases}$$

After normalization it is easily found that the normalized weights $\bar{w}_i^* \neq \hat{w}_i^*$, $i = 1, \dots, n$. □