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Center conditions for nilpotent cubic systems using the Cherkas method☆

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Abstract

In this study, we consider the center problem of a cubic polynomial differential system with a nilpotent linear part. The analysis is based on the application of the Cherkas method to the Takens normal form. The analysis requires many computations, which are verified by employing one algebraic manipulator and extensive use of the computer algebra system called Singular.

Keywords: Nilpotent center problem; Analytic integrability; Cherkas method; Decomposition in prime ideals; Takens normal form

1. Introduction and preliminary results

We consider the following differential system with a nilpotent linear part

$$\dot{x} = y + P(x, y), \qquad \dot{y} = Q(x, y), \tag{1.1}$$

where P and Q are analytic in a neighborhood of the origin without constants as linear terms. We assume that the origin is an isolated singular point. The *monodromy problem* involves characterizing when a singular point is either a focus or a center. Andreev [4] solved this problem for nilpotent singular points.

Theorem 1.1 (*Andreev*). *Let $y = \phi(x)$ be the solution of $y + P(x, y) = 0$ that passes through the origin. Consider the functions*

$$\psi(x) = Q(x, \phi(x)) = a_\alpha x^\alpha + \mathcal{O}(x^{\alpha+1})$$
$$\Delta(x) = \mathrm{div}(P, Q)(x, \phi(x)) = b_{\tilde{\beta}} x^{\tilde{\beta}} + \mathcal{O}(x^{\tilde{\beta}+1}),$$

with $a_\alpha \neq 0, \alpha \geq 2$ and $b_{\tilde{\beta}} \neq 0, \tilde{\beta} \geq 1$, or $\Delta(x) \equiv 0$. Then, the origin of (1.1) is monodromic if and only if $a_\alpha < 0, \alpha = 2\tilde{n} - 1$ is an odd number and one of the following conditions holds: (i) $\tilde{\beta} > \tilde{n} - 1$; (ii) $\tilde{\beta} = \tilde{n} - 1$ *and* $b_{\tilde{\beta}}^2 + 4\tilde{n} a_\alpha < 0$; (iii) $\Delta(x) \equiv 0$.

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The *Andreev number* n of a monodromic singular point at the origin of system (1.1) is the integer $\tilde{n} \geq 2$ given in Theorem 1.1. In fact this Andreev number n is invariant under analytic (formal) orbital conjugations of system (1.1) (see [15]). However, when (1.1) is analytically conjugate or analytically orbitally conjugate to another differential system, $\tilde{\beta}$ generally changes although the monodromic relation $\tilde{\beta} \geq \tilde{n} - 1$ remains invariant (see [15,17]).

First, we recall some known results regarding the normal forms of nilpotent singular points. If the origin of system (1.1) is monodromic, then by changing $(x, y) \to (x, y - \phi(x))$ and scaling the axes $(x, y) \to (sx, -sy)$ with $s = (-1/a_\alpha)^{1/(2-2\tilde{n})}$, system (1.1) is transformed into the pre-normal form

$$\dot{x} = y(-1 + X_1(x, y)) \qquad \dot{y} = f(x) + y\delta(x) + y^2 Y_0(x, y), \tag{1.2}$$

where $X_1(0, 0) = 0$, $f(x) = x^{2\tilde{n}-1} + \cdots$ with $\tilde{n} \geq 2$ and either $\delta(x) \equiv 0$ or $\delta(x) = b_{\tilde{\beta}} x^{\tilde{\beta}} + \cdots$ with $\tilde{\beta} \geq \tilde{n} - 1$ (see [15,17]).

Takens [33] proved that a system with a nilpotent singular point at the origin can be formally transformed into a generalized Liénard system

$$\dot{x} = -y, \qquad \dot{y} = a(x) + y\tilde{b}(x), \tag{1.3}$$

where $a(x) = a_s x^s (1 + \mathcal{O}(x))$ with $s \geq 2$ and $\tilde{b}(x)$ with $\tilde{b}(0) = 0$ are formal power series. Later, Strózyna and Zoladek [32] proved that this normal form can be achieved through an analytic change of variables. Moreover, in the monodromic case, $s = 2n - 1$ with $n \geq 2$, and after changing $x \mapsto u$ with $u(x) = (2n \int_0^x a(z)dz)^{1/(2n)} = x(a_{2n-1} + \mathcal{O}(x))^{1/(2n)}$ and the reparameterization of time $t \mapsto \tau$ with $dt/d\tau = u^{2n-1}/a(x) = a_{2n-1}^{-1/(2n)} + \mathcal{O}(x)$, we can simplify the normal form above into

$$\dot{x} = -y, \qquad \dot{y} = x^{2n-1} + yb(x), \tag{1.4}$$

where $b(x)$ is an analytic function obtained from $a(x)$ and $\tilde{b}(x)$ of the form $b(x) = \sum_{i \geq \beta} b_i x^i$. From system (1.4), we can characterize the centers of monodromic nilpotent singularities under the condition that $b(x)$ has to be an odd function, as shown by the following result.

Theorem 1.2 (*Moussu*). *Consider the analytic system (1.4) where its origin is a monodromic critical point, i.e., it satisfies one of the following conditions:* (i) $\beta > n - 1$*;* (ii) $\beta = n - 1$ *and* $b_\beta^2 - 4n < 0$*;* (iii) $b(x) \equiv 0$*. Then, the origin is a center if and only if* $b(x)$ *is an odd function.*

Consequently, all of the nilpotent centers are analytically orbitally reversible. This result was given by Moussu without using the analyticity of the change that transforms system (1.1) into the normal form (1.4) (see [29]). In [29], it was also proved generically that the nilpotent centers do not have an analytic first integral in the neighborhood of a nilpotent singularity (also see [7]). However, there are examples of families of nilpotent centers with an analytic first integral (see [7,8,14,20,21,17,24]). For instance, the nilpotent centers of system (1.1), where P and Q are homogeneous polynomials of the same odd degree, always have an analytic first integral (see [3,5]). A generalization of this family inside differential systems that are sums of quasi-homogeneous polynomials was given by [1] (also see [14]). In fact, the existence of the analytic first integral for this family was proved by [14], while [1] proved that this family has a C^∞ Lyapunov function.

As shown in the next sections, it is not necessary to obtain the complete normal form for practical use of the normal form (1.4). It is sufficient to write the system in the pre-normal form

$$\dot{x} = -y + \mathcal{O}(|(x, y)|^r), \qquad \dot{y} = x^{2n-1} + yb_r(x) + \mathcal{O}(|(x, y)|^r), \tag{1.5}$$

for suitable r, and the polynomial $b_r(x)$ must be an odd function to have a center.

From the normal form (1.4) and the pre-normal form (1.5), we can derive an algorithm to compute the focal values of a monodromic nilpotent singular point. However, this approach is computationally expensive, even when determining the stability (the first non-null focal value) of a nilpotent singular point. Thus, new methods for computing the focal values are of interest. Several methods have been developed in recent decades for detecting nilpotent centers and [23] provided a short review of them.

The oldest method is the characterization of a nilpotent center through the Poincaré return map using the Liapunov generalized coordinates (see [6]). Using Liapunov generalized coordinates is a very difficult problem, so other methods