Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 129 (2016) 69–80
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original articles

Determining surface heat flux for noncharacteristic Cauchy problem for Laplace equation

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Received 9 January 2012; received in revised form 12 July 2013; accepted 25 April 2016
Available online 9 May 2016

Abstract

In this paper, the noncharacteristic Cauchy problem for the Laplace equation

$$\begin{cases} w_{xx} + w_{yy} = 0 & x \in (0, 1), \ y \in R, \\ w(0, y) = g(y) & y \in R, \\ w_x(0, y) = h(y) & y \in R, \end{cases}$$

is investigated, where the Cauchy data is given at $x = 0$ and the heat flux is sought in the interval $0 < x \leq 1$. This problem is severely ill-posed: the solution (if it exists) does not depend continuously on the given data. A modified regularization method is used to solve this problem. Furthermore, some error estimates for the heat flux between the regularization solution and the exact solution are given. Finally, a numerical example shows that the proposed method works well.

Keywords: Ill-posed problem; Cauchy problem; Laplace equation; Modified regularization method; Error estimate

1. Introduction

In many physics and engineering problems, we need to determine the temperature or heat flux on the surface of a body from internal measurement [4]. The Cauchy problem for the Laplace equation is equal to a two-dimensional steady state heat conduction equation, which arises in a lot of practical applications, including cardiology [9], non-destructive testing [1], bioelectric field problem [15], seismology [26], geophysics [18] and plasma physics [12].

To our knowledge, so far there are many articles on the following Cauchy problem for the Laplace equation

$$\begin{cases} w_{xx} + w_{yy} = 0 & x \in (0, 1), \ y \in R, \\ w(0, y) = g(y) & y \in R, \\ w_x(0, y) = 0 & y \in R, \end{cases} \tag{1}$$

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http://dx.doi.org/10.1016/j.matcom.2016.04.009
0378-4754/

where one wants to determine the temperature $w(x,\cdot)$ and the heat flux $w_x(x,\cdot)$ for the Cauchy data $w(0,y)=g(y)$ and $w_x(0,y)=0$. However, the data $w_x(0,y)=0$ is too strict as usual. In this paper, we consider the following noncharacteristic Cauchy problem for the Laplace equation as follows:

$$\begin{cases} w_{xx}+w_{yy}=0 & x\in(0,1),\ y\in R, \\ w(0,y)=g(y) & y\in R, \\ w_x(0,y)=h(y) & y\in R. \end{cases} \tag{2}$$

Here, we want to determine the heat flux for $w_x(x,\cdot)$ for $0<x\leq 1$ from the Cauchy data $w(0,y)=g(y)$ and $w_x(0,y)=h(y)$.

It is well-known that the Cauchy problem for the Laplace equation is severely ill-posed in the sense of Hadamard, i.e., the solution (if it exists) does not depend continuously on the given Cauchy data. In other words, a small error in the given Cauchy data can cause a large error in the solution. Thus, several regularization methods are needed.

In recent years, plenty of regularization methods have been used for this problem, for instance, quasi-reversibility method [16,17,23], Tikhonov regularization method [2,6,25], Fourier regularization method [5,11], conjugate gradient method [13], the boundary element method [19], finite element method [3], moment method [8,14], and central difference method [28]. In this paper, we modify the equation to obtain a stable approximation, which we learned from Eldén [10]. In [10], Eldén discussed a standard inverse heat conduction problem and the idea firstly came from Webber [27]. The modified regularization method has been studied for several inverse problems, such as, backward heat conduction problem [21,29], high order numerical derivatives [20] and inverse heat conduction problem [24,22].

The rest of this paper is organized as follows. In Section 2, we demonstrate ill-posedness of Cauchy problem for the Laplace equation and introduce the modified regularization method. In Section 3, error estimates between the regularization solution and the exact solution are given under a priori choice of the regularization parameter. In Section 4, a numerical example is given. Finally, we conclude the paper in Section 5.

2. Ill-posedness of Cauchy problem of Laplace equation and regularization

Consider the following Cauchy problems for the Laplace equation:

$$\begin{cases} u_{xx}+u_{yy}=0 & x\in(0,1),\ y\in R, \\ u(0,y)=g(y) & y\in R, \\ u_x(0,y)=0 & y\in R, \end{cases} \tag{3}$$

and

$$\begin{cases} v_{xx}+v_{yy}=0 & x\in(0,1),\ y\in R, \\ v(0,y)=0 & y\in R, \\ v_x(0,y)=h(y) & y\in R. \end{cases} \tag{4}$$

Our aim is to seek the solutions $u_x(x,y)$ and $v_x(x,y)$ from the given data $g(y)$, $h(y)$, respectively. In fact, $g(y)$, $h(y)$ cannot be measured exactly, so we would actually have measurement functions $g^\delta(\cdot), h^\delta(\cdot)\in L^2(R)$, satisfying

$$\|g^\delta-g\|\leq\delta, \qquad \|h^\delta-h\|\leq\delta, \tag{5}$$

where the constant δ represents a error bound, $\|\cdot\|$ denotes the L^2-norm, and there exists a finite positive constant E, and further that the following a priori bound exists

$$\|u(1,\cdot)\|\leq E, \qquad \|v_x(1,\cdot)\|\leq E. \tag{6}$$

Here, $\delta\ll E$. Thus, problem (2) can be solved because $w_x(x,y)=u_x(x,y)+v_x(x,y)$.

In order to determine the flux $w_x(x,y)$ for $0<x\leq 1$ from the given data $g(\cdot)=w(0,\cdot)$, $h(\cdot)=w_x(0,\cdot)$, when $w(x,y)$ satisfies

$$\begin{cases} w_{xx}+w_{yy}=0 & x\in(0,1),\ y\in R, \\ w(0,y)=g(y) & y\in R, \\ w_x(0,y)=h(y) & y\in R. \end{cases}$$