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On the pseudolocalized solutions in multi-dimension of Boussinesq equation☆

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Abstract

A new class of solutions of three-dimensional equations from the Boussinesq paradigm are considered. The corresponding profiles are not localized functions in the sense of the integrability of the square of the solution over an infinite domain. For the new type of solutions, the gradient and the Hessian/Laplacian are square integrable. In the linear limiting case, analytical expressions for the profiles of the pseudolocalized solutions are found. The nonlinear case is treated numerically with a special approximation of the differential operators with spherical symmetry that allows for automatic acknowledgement of the behavioral conditions at the origin of the coordinate system. The asymptotic boundary conditions stem from the $1/\rho$ behavior at infinity of the pseudolocalized profile. A special approximation is devised that allows us to obtain the proper behavior with a much smaller computational domain. The pseudolocalized solutions are obtained for both quadratic and cubic nonlinearity.

Keywords: Pseudolocalized solutions; Quasi-particles; Peakons

1. Introduction: localized waves and quasi-particles

Russell [23] observed the permanent (or “great”) wave that can travel large distances without changing form. In 1871–1872 Boussinesq came up with a fundamental idea: the balance between the nonlinearity and dispersion is what makes the shape of the wave permanent [1,2]. He found the first analytical solution of *sech* type. Later on, Korteweg and de Vries [19] showed that in a frame moving with the characteristic speed of the wave, the second order in time Boussinesq equation can be reduced to a first-order in time equation (known now as KdV equation) for which the balance between nonlinearity and dispersion holds as it does for its parent (the Boussinesq) equation. Boussinesq-type equations arise not only in shallow water flows but also in the theory of shells and plates (the famous von Kármán equations) [18,15]. Even the Schrödinger equation of wave mechanics can be shown to be a Boussinesq equation for the real or imaginary part of the wave function [8]. Boussinesq equation is a generalized wave equation containing dispersion in the form of a biharmonic operator of the sought function. The dispersive effects of the biharmonic operator can be countered by the presence of a nonlinearity, and as a result, a permanent wave of localized type may exist and propagate without change. We call this balance between the nonlinearity and dispersion the “Boussinesq paradigm”.

☆ Guest Editor’s Note: Prior to the acceptance of the revised manuscript, the author of this article passed away. As a result, all proof-corrections were made by Dr. Ivan C. Christov (E-mail: christov@alum.mit.edu). All further correspondence should be addressed to him.

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One of the main properties of "permanent" localized waves is the collision property. Skyrme [24] argued the idea that different particles are the localized waves of different field equations and actually found a particle-like behavior of the localized solution of the *sine*-Gordon equation [20]. Unfortunately, the latter solution was kinks ('hydraulic jumps') which did not appeal directly to the physical intuition of particles being localized 'lumps' of energy. Actually the works of Skyrme defined in essence the notion of soliton which was introduced a couple of years later by Zabusky and Kruskal [27] who discovered numerically that the solitary waves of KdV equation retain their shapes after multiple collisions. They introduced the coinage *soliton* to emphasize this particle-like behavior. Currently the term *soliton* in the strict sense is reserved only for the particle-like localized solutions of fully integrable systems. For integrable and nonintegrable systems alike, the evolution of a wave profile which is a superposition of two localized waves results in a virtual recovery of the shapes of the two initial localized waves, but with shifted positions, outgoing from the site of interaction. The collision property gave rise to the term "quasi-particle" (or QP). When the physical system is described by equation(s) that conserve the energy and momentum and some integral interpreted as the pseudomass of a localized shape, the quasi-particles behave very much as actual particles in quantum physics.

Yet there are essential differences between the quasi-particles currently known and the real particles. The most conspicuous is that the research is mostly concerned with 1D cases, which have virtually no relevance to the physical reality. Even within the class of 1D QPs there are more questions to be answered before they are shown to qualify as particles. One of the unsatisfactory features is that the quasi-particles appear to pass through each other (they 'percolate', in a sense, through each other) rather than scatter. This does not prevent investigators from speaking about scattering of solitons. To the limit of author's knowledge, the only work that attempts to answer the question of whether the *sine*-Gordon kinks percolate or scatter is [16,14], where the Variational Approximation is applied. The standard approach without additional collective variables (see [13], and literature cited therein) cannot answer this question.

It has been recently discovered (see [26] and further corroboration in [25]) that under certain conditions the solitons of the System of Coupled Nonlinear Schrödinger Equations (SCNLSE) (called alternatively Vector NLSE) can actually scatter without crossing each other's paths (percolating through each other).

1.1. Classical solitons as quasi-particles: the Con's

Mathematicians place emphasis on the full integrability of the system that exhibits soliton solutions. However, physically speaking, the three laws known to govern motion are the conservation of mass, energy, and momentum. The full integrability is not directly relevant for the physical implication of localized solutions as quasi-particles.

There are essential differences between the real particles and 1D quasi-particles known currently for which hundreds of strict mathematical results are proven. *First*, some of the known quasi-particles may pass through each other (percolate in a sense), i.e. they do not repel (scatter) like the real particles. *Second*, the repulsion or attraction of the QPs depends very much on the potential of interaction defined by the asymptotic behavior of their tails. In 1D, for virtually all main soliton supporting equations the potential is attractive, and decays exponentially with the distance between the QPs. In physics, at large distances the potential (producing the gravitation force) decays algebraically as the inverse of the distance, which is a radical difference from an exponential decay. *Third*, results about the interaction in more than one dimension are rare and mostly qualitative, not quantitative.

The above list of discrepancies between the currently known QPs and the actual physical particles can be extended, but the main points are those mentioned above.

1.2. The first forays in the realm of 2D Boussinesq solitons

The first thing that comes to mind is that this exponentially decaying asymptotic behavior is due to the one-dimensional nature of the QPs. Unfortunately, very little is known about solitary waves that are truly localized in two spatial variables. For the time being, the main success has been in finding solitary waves of the Kadomtsev–Petviashvili (KP) equation (see the original article [17], also [12,21,22] and the literature cited therein).

The strictly localized steadily propagating solutions of truly 2D equations of Boussinesq type were not investigated until recently. The question is very important and we have developed three different numerical techniques to interrogate the problem. A fast Galerkin spectral method is used in [6]; a semi-analytical method for relatively small propagation speed is presented in [10]; and a finite-difference solution with special implementation of the asymptotic boundary