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Original Article

On the geometry of the rotating liquid drop

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Abstract

Here we consider the problem of a fluid body rotating with a constant angular velocity and subjected to surface tension. Determining the equilibrium configuration of this system turns out to be equivalent to the geometrical problem of determining the surface of revolution with a prescribed mean curvature. In the simply connected case, the equilibrium surface can be parameterized explicitly via elliptic integrals of the first and second kind. Here, we present two such parameterizations of the drops and we use the second of them to study finer details of the drop surfaces such as the existence of closed geodesics.

Keywords: Profile curves; Surfaces; Curvatures; Geodesics

1. Introduction

Let us consider a fluid body with incompressible mass rotating with a constant angular velocity and subjected to surface tension. The problem then is to find the shape which this mass will have under prescribed angular velocity.

The only force acting inside the drop is the centripetal force generated from the potential $\rho\omega^2 r^2/2$ and directed away from the axis. Here, ρ is the fluid density, ω is the fixed angular velocity and r is the radial distance from the axis of rotation. Because the fluid is assumed to be incompressible, a constant internal pressure p_i results within the body. According to the Laplace–Young equation [9], at the free surface $\mathcal{S}$ enclosing the drop the surface tension σ generates a pressure proportional to its mean curvature H, σH. Finally, we must also add the constant pressure p_e of the external fluid. The surface in equilibrium is therefore described by equating these pressures:

$$p_i + \frac{\rho\omega^2 r^2}{2} = \sigma H + p_e, \tag{1}$$

where ω, p_e, p_i, ρ, σ are constants. This then reduces immediately to the equation:

$$H = 2\tilde{a}r^2 + \tilde{c}, \tag{2}$$

where $\tilde{a}$ is a positive constant and $\tilde{c}$ is arbitrary.

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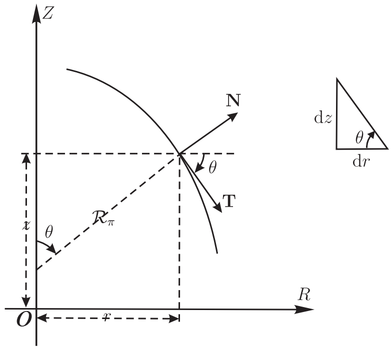

Fig. 1. Geometry of the profile curve.

2. Geometry and surface invariants

Let us now recall the well-known fundamental relations among meridional curvature κ_μ, circumferential curvature κ_π and mean curvature H for surfaces of revolution [3],

$$\kappa_\mu = \frac{\mathrm{d}(r\kappa_\pi)}{\mathrm{d}r}, \quad H = \frac{\kappa_\mu + \kappa_\pi}{2}. \tag{3}$$

The simultaneous solution of the system of Eqs. (2) and (3) is

$$\kappa_\pi = \tilde{a}r^2 + \tilde{c} + \frac{C}{r^2}, \quad \kappa_\mu = 3\tilde{a}r^2 + \tilde{c} - \frac{C}{r^2}, \tag{4}$$

where C is an integration constant which will be assumed to be zero from now on. We make this assumption here in order to get analytical formulas for the rotating drop. At the moment, letting C be non-zero only affords numerical analysis.

Moreover, if $\mathcal{R}_\pi$ is the distance from the surface point to the intersection of the normal at that point with the symmetry axes (see Fig. 1, where we use $\mathcal{R}$ for the abscissa simply to match the notation $\mathcal{R}_\pi$), then we have additionally,

$$\kappa_\pi = \frac{1}{\mathcal{R}_\pi} = \frac{\sin\theta}{r}, \tag{5}$$

and therefore

$$\frac{\mathrm{d}z}{\mathrm{d}r} = -\tan\theta = -\frac{r\kappa_\pi}{\sqrt{1-(r\kappa_\pi)^2}}. \tag{6}$$

Note that we take the convention that θ is measured from the horizontal line through the point and is positive when going clockwise. This means that θ will begin negative with the consequence that the derivative $\mathrm{d}z/\mathrm{d}r$ will be positive and the height of the curve will increase.

Finally, the ordinate of the profile curve is given by the integral

$$z(r) = -\int \frac{(\tilde{a}r^3 + \tilde{c}r)\mathrm{d}r}{\sqrt{1-(\tilde{a}r^3+\tilde{c}r)^2}}. \tag{7}$$

Here, the indefinite integral produces an expression in r that gives the r-dependent formula for z. From these considerations, we obtain a profile curve for the rotating drop, $(r, z(r))$ with

$$\mathbf{x}(u, v) = (r\cos(v), r\sin(v), z(r)),$$

being a parametrization for the surface of revolution $\mathcal{S}$ that is the drop itself.