Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 122 (2016) 55–68
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original articles

Partial chaos suppression in a fractional order macroeconomic model

S.A. David[a,*], J.A.T. Machado[b], D.D. Quintino[c], J.M. Balthazar[d]

[a] University of São Paulo at Pirassununga, Av. Duque de Caxias Norte, 225 – 13635-900, Pirassununga, SP, Brazil
[b] Institute of Engineering, Polytechnic of Porto, Rua Dr. Antonio B. de Almeida 431 – 4249-015, Porto, Portugal
[c] Federation of Industries of the State of São Paulo, Av. Paulista, 1313 – 01311-923, São Paulo, SP, Brazil
[d] State University of São Paulo at Rio Claro, Avenida 24 A, 1515- 13506-900, Rio Claro, SP, Brazil

Received 7 August 2015; received in revised form 18 October 2015; accepted 24 November 2015
Available online 11 December 2015

Abstract

This work investigates the possibility of suppressing chaos in a fractional-nonlinear macroeconomic dynamic model. The system generalizes a model recently reported in the literature in which chaos is strongly present. This description involves the inclusion of the public sector deficit and its coupling with other variables. The system is simulated for integer and non-integer orders that produce a complex dynamics. The time histories and the phase diagrams are presented. The main contribution of this work refers to the adoption of the largest Lyapunov exponent (LLE) criteria based on Wolf's algorithm. This approach improves the response of the system, suppressing, at least partially, the strong presence of chaos reported in previous studies.

Keywords: Macroeconomic model; Fractional calculus; Numerical simulations; Nonlinear systems; Chaos

1. Introduction

The dynamics of markets require an accurate modeling for understanding the complex behavior of real world financial and economic systems. A frequent factor in dynamic nonlinear systems is *path dependence*, so that, the history leading to a supposed level of equilibrium may neither be unique, nor predetermined. This behavior contrasts with the concept of equilibrium in neoclassical economics, in which output presumably reaches equilibrium regardless of its starting point or even in the presence of temporary crisis. In *path dependence*, the starting conditions, or some noise in the trajectory, can have significant effects on the final outcome, and possibly irreversible consequences.

Studies proposed recently employ fractional derivatives in financial and economic modeling. However, they do not consider the public sector deficit variable in the model and the obtained results indicate a strong presence of chaos.

In this study, we investigate possible impacts resulting from the introduction of an additional variable, namely, the public sector deficit. A fractional differential equation is introduced and their coupling with system dynamic equations is considered. Consequently, one can see how changes in the functional form of the model may lead to different

* Corresponding author. Tel.: +55 19 35656711.
E-mail addresses: sergiodavid@usp.br (S.A. David), jtm@isep.ipp.pt (J.A.T. Machado), derick.quintino@fiesp.org.br (D.D. Quintino), jmbaltha@rc.unesp.br (J.M. Balthazar).

http://dx.doi.org/10.1016/j.matcom.2015.11.004
0378-4754/

expansion paths in economics [31]. It is noteworthy that numerous studies about financial–economic markets have been published. However, merely a small part of them use FC as a modeling.

FC can represent systems with high-order dynamics and complex nonlinear phenomena using just a few coefficients, since the arbitrary order of the derivatives provides an additional degree of freedom to fit a specific behavior [26]. In fact, the importance of studies involving nonlinear dynamic and chaos in economic was demonstrated in recent years. This property, allied to the fact that financial–economic variables possess long memories [25], motivated our choice for a more appropriate nonlinear fractional modeling. This paper is organized as follow: in Section 2, we present a brief literature review. In Section 3, we present the fractional order model and several considerations about inclusion of sector public deficit. In Section 4, we consider a sensibility analysis in addition to show and discuss about the numerical simulation results of the fractional order model. Finally, in Section 5 we outline the main concluding comments.

2. A brief literature review

Financial and economic variables, such as foreign exchange rates, gross domestic product, interest rates, production, and stock market prices reveal long memory. Indeed, correlations overlap with the longest time scales in the financial–economic market [25,34]. This means that all past fluctuations in financial and economic variables present correlations towards the future. With this fact in mind, FC [8,10,19,24] can be a convenient tool in order to deal with that property. Investigations using FC in financial–economic systems were conducted in [7,12,15,21,20,23,28–30].

Continuous nonlinear and chaotic models that have been proposed to study complex economic dynamics, namely, the forced Van der Pol model [6], the IS–LM (Investment Saving–Liquidity Money) model [3] and others [1,5,13, 14,22,37]. Ma and Chen [17,18] reported an interesting dynamic model of economic system. They found that the model provides irregularity and extreme sensitivity to the initial value of the state and the parameters. Chen [4] considered a generalization of the system proposed in [17,18] for fractional orders. Two typical routes to chaos – period doubling and intermittency – were reported. Li and Peng [16] showed a strong presence of chaos in Chen's system with fractional order. However, Chen's model does not consider public sector deficit in the dynamic equations. David et al. [9] proposed a model of fractional order involving the public sector deficit, but the results are inconclusive about the presence (or not) of chaos.

3. Fractional order macroeconomic model

In this work, it is considered the public sector deficit W and its coupling with other variables. Furthermore, it is investigated the Largest Lyapunov Exponent (LLE) [2,27] in order to provide an assertive response about the presence of chaotic behavior.

The Lyapunov exponents [11,36] evaluate the sensitive dependence on initial conditions by considering the exponential divergence of nearby orbits. Therefore, one needs to evaluate how trajectories with nearby initial conditions diverge, since they are related to the expanding and contracting nature of different directions in phase space.

The dynamics of the system transform the D-sphere of states in to D-ellipsoid; therefore, when a chaotic motion emerges, a complex evolution exists. The instabilities are associated with the directions where the stretching occurs, while the stability is associated with the contraction directions.

Mathematically, the Lyapunov exponents consider $d(t) = d_0 b^{\lambda t}$, where b is a reference basis. In this perspective, the following definition is considered,

$$\lambda_i = \lim_{t\to\infty} \left(\frac{1}{t} \log_b \left(\frac{d_i(t)}{d_0} \right) \right), \quad i = 1, 2, \ldots, D, \tag{1}$$

where $d_i(t)$ is the deformed hyper-volume in time instant t. The signs of the Lyapunov exponents provide information about the system's dynamics and the greater exponent is an important index to diagnose chaotic motion.

The model adopted here describes the time-variation of four state variables: the interest rate, X, the investment demand, Y, the price index, Z and the public sector deficit, W. The parameters, a, b, c and d, are nonnegative, so that a is the saving amount, b is the cost per investment, c is the elasticity of demand of commercial markets and d represents the cost of public debt.