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Weak approximation of Heston model by discrete random variables

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Abstract

We construct a first-order weak split-step approximation for the solution of the Heston model that uses, at each step, generation of two discrete two-valued random variables. The Heston equation system is split into the deterministic part, solvable explicitly, and the stochastic part that is approximated by discrete random variables. The approximation is illustrated by several simulation examples, including applications to option pricing.

Keywords: Heston model; CIR; Weak approximations; Split-step approximations; Option pricing

1. Introduction

We consider the solution of the widely used stochastic volatility model proposed by Heston [10]:

$$\begin{cases} dS_t = rS_t dt + \sqrt{Y_t} S_t d\widetilde{W}_t, \quad S_0 = s \geq 0, \\ dY_t = k(\theta - Y_t)dt + \sigma\sqrt{Y_t} dW_t, \quad Y_0 = y \geq 0, \\ dW_t \, d\widetilde{W}_t = \rho dt, \end{cases} \tag{1.1}$$

where W and $\widetilde{W}$ are (possibly, dependent) standard Brownian motions, with parameters $\theta, \sigma, k > 0$.

Similarly to the CIR model, one of the problems with constructing efficient numerical methods for the Heston model is the square-root coefficients. Although, theoretically, the distribution (the transition density) of the CIR process is known, for simulation, it is more convenient to use numerical methods for solving stochastic differential equations (SDEs); see, for example, [1,2,13,14,16] and references therein. For the Heston model, this seems to be even indispensable since the distribution of the Heston process, S in Eq. (1.1), is explicitly known only in the form of the characteristic function of $X_t = \log S_t$ [10,5].

Classical approximation methods do not provide any useful solution. For example, the Euler approximation, although in "good" cases is a first-order weak approximation, takes negative values with positive probability, and

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various attempts to modify the approximation to make it positive (see [12]) seem to produce approximations converging rather slowly.

The Heston model is known for almost 20 years. Due to its popularity, nowadays one has a variety of approximations of the Heston process to choose from (see, e.g., [3,12] for a summary of different approaches in the field). In this paper, we propose a new scheme that only uses, at each step, generation of two discrete two-valued random variables. It extends the idea of [16] (used for approximation of the CIR process) of using only simple discrete variables, together with combining split-step (see, e.g., [17,2] and [15] for CIR equations in the Stratonovich and Itô forms, respectively) and moment matching (e.g., [3]) techniques. The advantage of such a scheme is that, in comparison with other known schemes (such as Andersen's QE [3] and Alfonsi [2] schemes), it is (i) much simpler to implement (see Section 5) and (ii) has a significantly lower computational cost (see Section 6, Table 3) since it only needs a single uniform random variable in each generation step; at the same time, it provides a satisfactory accuracy.

The paper is organized as follows. After some preliminaries and definitions in Section 2, in Section 3, we "split" the approximation problem for the process (S, Y) in Eq. (1.1) into exact solution of the deterministic part and the approximation problem for the stochastic part of the system. In Section 4, we construct a potential first-order weak approximation for the stochastic part. We summarize the algorithm in Section 5. In Section 6, we illustrate the scheme by numerical simulation results, including option pricing and a detailed comparison with the schemes of Andersen [3] and Alfonsi [2]. The Appendix contains proofs of two auxiliary lemmas used in the proof of the main theoretical result, Theorem 4.1.

2. Preliminaries and definitions

Consider the general two-dimensional SDE

$$dZ_t = b(Z_t, t)dt + \sigma(Z_t, t)\,dB_t, \quad t \geq 0,\ Z_0 = z, \tag{2.1}$$

for $z \in \mathbb{D} \subset \mathbb{R}^2$ with standard two-dimensional Brownian motion $B_t = (B_t^1, B_t^2)$ and coefficients $b : \mathbb{R}^2 \times I \to \mathbb{R}^2$ and $\sigma = (\sigma_{ij}), i, j = 1, 2$, where $\sigma_{ij} : \mathbb{R}^2 \times I \to \mathbb{R}^2$. We assume that the SDE is domain-preserving in the sense that, for every $z \in \mathbb{D}$, the SDE has a unique weak solution Z^z such that as $P\{Z_t^z \in \mathbb{D},\ t \geq 0\} = 1$.

On a fixed time interval $[0, T]$, we consider equidistant time discretizations $\Delta^h = \{ih, i = 0, \dots, [T/h]\}$, where $[a]$ denotes the integer part of a number a.

By $C_0^\infty(\mathbb{D})$ we denote the functions on $\mathbb{D}$ of class C^∞ with compact support, and by $C_{pol}^\infty(\mathbb{D})$ the real-valued functions of class C^∞ with all partial derivatives of polynomial growth:

$$C_{pol}^\infty(\mathbb{D}) := \left\{ f \in C^\infty(\mathbb{D}) : \forall i \in \mathbb{N}_0^2,\ \exists C_i > 0, \exists k_i \geq 0,\ \forall z \in \mathbb{D}, |f^{(i)}(z)| \leq C_i(1 + |z|^{k_i}) \right\}.$$

Here $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, $i = (i_1, i_2) \in \mathbb{N}_0^2$ are multiindices, and

$$f^{(i)}(z) := \frac{\partial^{|i|} f(z)}{\partial z^i}, \quad |i| := i_1 + i_2,\ \partial z^i := \partial z_1^{i_1} \partial z_2^{i_2}.$$

Following [2], we call the "sequence" $\{(C_i, k_i) : i \in \mathbb{N}_0^2\}$ a good sequence for f.

Definition 2.1. A discretization scheme $\overline{Z}^h$ is a family of discrete-time $\mathbb{D}$-valued time-homogeneous Markov chains $\overline{Z}^h = \{\hat{Z}^h(z, t),\ z \in \mathbb{D},\ t \in \Delta^h\},\ h > 0$, with initial values $\overline{Z}^h(z, 0) = z$.

With some abuse of notation, sometimes we also write $\overline{Z}_t^z$ or $\overline{Z}(z, t)$ instead of $\overline{Z}^h(z, t)$. Note that to define a discretization scheme (in distribution), it suffices to construct one-step "transitions," that is, the (distributions of) random variables $\overline{Z}_h^z = \overline{Z}^h(z, h)$ for all $z \in \mathbb{D}$.

Definition 2.2. A discretization scheme $\overline{Z}^h$ is a weak υth-order approximation for the solution Z^z of Eq. (2.1) if, for every $f \in C_0^\infty(\mathbb{D})$, there exists $K > 0$ such that

$$\left|\mathbb{E}f(Z_T^z) - \mathbb{E}f(\overline{Z}_T^z)\right| = \left|\mathbb{E}f(Z_T^z) - \mathbb{E}f\left(\overline{Z}^h(z, T)\right)\right| \leq Kh^\upsilon, \quad h > 0. \tag{2.2}$$