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Integer rounding and modified integer rounding for the skiving stock problem

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HIGHLIGHTS

- For the first time: profound theoretical investigations on the gap of the SSP.
- Generalization of Zak's theorem to arbitrary values of m.
- Proof of the MIRDP for the divisible case of the SSP.
- Construction of an infinite number of non-equivalent non-IRDP instances.
- We show how a gap arbitrarily close to 22/21 can be obtained.

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ABSTRACT

We consider the one-dimensional skiving stock problem which is strongly related to the dual bin packing problem: find the maximum number of items with minimum length L that can be constructed by connecting a given supply of $m \in \mathbb{N}$ smaller item lengths $l_1, \ldots, l_m$ with availabilities $b_1, \ldots, b_m$. For this optimization problem, we investigate the quality of the continuous relaxation by considering the gap, i.e., the difference between the optimal objective values of the continuous relaxation and the skiving stock problem itself. In a first step, we derive an upper bound for the gap by generalizing a result of E. J. Zak. As a main contribution, we prove the modified integer round-down property of the divisible case. In this context, we also present a construction principle for non-IRDP instances of the divisible case that leads to gaps arbitrarily close to 22/21.

1. Introduction and preliminaries

In this paper, we consider the one-dimensional skiving stock problem (SSP) [1,2] which is strongly related to the dual bin packing problem (DBPP) in literature (see e.g. [3,4] or [5]). In the classical formulation, $m \in \mathbb{N} := \{1, 2, \ldots\}$ different item lengths $l_1, \ldots, l_m$ with availabilities $b_1, \ldots, b_m$ are given, the so-called *item supply*. We aim at maximizing the number of products with minimum length L that can be constructed by connecting the items on hand.

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Such computations are of high interest in many real world applications, e.g. industrial production processes (see [2] for an overview) or politico-economic problems (cf. [3,4]). Furthermore, also neighboring tasks, such as dual vector packing problem [6] or the maximum cardinality bin packing problem [7,5], are often associated or even identified with the dual bin packing problem. These formulations are of practical use as well since they are applied in multiprocessor scheduling problems [8] or surgical case plannings [9].

Throughout this paper, we will use the abbreviation $E := (m, l, L, b)$ for an instance of the SSP with $l = (l_1, \dots, l_m)^\top$ and $b = (b_1, \dots, b_m)^\top$. Without loss of generality, we assume all input-data to be positive integers with $L > l_1 > \dots > l_m > 0$.

The classical solution approach is due to [2] and based on the formulation of Gilmore and Gomory in the context of one-dimensional cutting [10]. Any feasible arrangement of items leading to a final product of minimum length L is called *(packing) pattern* of E. We always represent a pattern by a nonnegative vector $a = (a_1, \dots, a_m)^\top \in \mathbb{Z}_+^m$ where $a_i \in \mathbb{Z}_+$ denotes the number of items of type $i \in I := \{1, \dots, m\}$ being contained in the considered pattern. For a given instance E, the set of all patterns is defined by $P_E := \left\{a \in \mathbb{Z}_+^m \,\middle|\, l^\top a \geq L\right\}$. In [11], the authors slightly improve Zak's formulation by only considering so-called minimal patterns obtaining a finite model, hereinafter referred to as the *standard model*, of the skiving stock problem. A pattern $a \in P_E$ is called *minimal* if there exists no pattern $\widetilde{a} \in P_E$ such that $\widetilde{a} \neq a$ and $\widetilde{a} \leq a$ hold (componentwise). The set of all minimal patterns is denoted by $P_E^\star$. Let $x_j \in \mathbb{Z}_+$ denote the number how often the minimal pattern $a^j = (a_{1j}, \dots, a_{mj})^\top \in \mathbb{Z}_+^m$ $(j \in J)$ of E is used, where $J = \{1, \dots, n\}$ represents an index set of all minimal patterns. Then the skiving stock problem can be formulated as

$$z^\star(E) = \max\left\{\sum_{j\in J} x_j \,\middle|\, \sum_{j\in J} a_{ij}x_j \leq b_i,\, i \in I,\, x_j \in \mathbb{Z}_+,\, j \in J\right\}.$$

A common (approximate) solution approach consists in considering the continuous relaxation

$$z_c^\star(E) = \max\left\{\sum_{j\in J} x_j \,\middle|\, \sum_{j\in J} a_{ij}x_j \leq b_i,\, i \in I,\, x_j \geq 0,\, j \in J\right\}$$

and the application of appropriate heuristics.

Practical experience and computational simulations, cf. [2], have shown that there is only a small gap $\Delta(E) := z_c^\star(E) - z^\star(E)$ for any instance E. Based on the contributions of Baum and Trotter [12] for general linear maximization problems, these observations have initiated the following definitions. A set $\mathcal{P}$ of instances has the *integer round-down property* (IRDP) if $\Delta(E) < 1$ holds for all $E \in \mathcal{P}$. An instance E with $\Delta(E) \geq 1$ is called *non-IRDP instance*. Furthermore, a set $\mathcal{P}$ of instances has the *modified integer round-down property* (MIRDP) if $\Delta(E) < 2$ is true for all $E \in \mathcal{P}$. It is conjectured in [2] that the one-dimensional skiving stock problem possesses the MIRDP.

In this paper, we investigate the gap of the skiving stock problem from a theoretical point of view. To this end, we first take the result of [2, Theorem 4], i.e., the proof of the IRDP for all instances with $m = 2$ item lengths, as an initial point and generalize this inequality to arbitrary values of m. This case and other special cases have also been studied by Marcotte [13] in the context of one-dimensional cutting.

In Section 3, we focus intensively on the *divisible case* of the skiving stock problem where $l_i | L$ (i.e. there is some $\gamma_i \in \mathbb{N}$ with $l_i/L = \gamma_i$) holds for each item length l_i $(i \in I)$. For these instances, we prove the MIRDP by means of the *first fit decreasing* (FFD) heuristic presented in [3] for the dual bin packing problem. Afterwards, we introduce a construction principle for an infinite number of non-equivalent non-IRDP instances of the divisible case. In a final step, we investigate the gap that can be obtained on the basis of these instances, give some conclusions, and provide an outlook of future research.