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Existence of Nash equilibrium for chance-constrained games

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ABSTRACT

We consider an n-player strategic game with finite action sets and random payoffs. We formulate this as a chance-constrained game by considering that the payoff of each player is defined using a chance constraint. We consider that the components of the payoff vector of each player are independent normal/Cauchy random variables. We also consider the case where the payoff vector of each player follows a multivariate elliptically symmetric distribution. We show the existence of a Nash equilibrium in both cases.

1. Introduction

In 1928, John von Neumann [19] showed that there exists a mixed strategy saddle point equilibrium for a two player zero sum matrix game. In 1950, John Nash [18] showed that there always exists a mixed strategy Nash equilibrium for an n-player general sum game with finite number of actions for each player. In both [18,19], it is considered that the players' payoffs are deterministic. However, there can be practical cases where the players' payoffs are better modeled by random variables following certain distributions and as a result players compete in a stochastic Nash game. The wholesale electricity markets are the good examples that capture this situation. The randomness in an electricity market is present due to various external factors, e.g., wind integration [17], and consumers' random demand [7].

One way to study stochastic Nash games is by using expected payoff criterion. Ravat and Shanbhag [21] considered stochastic Nash games using expected payoff criterion. They showed the existence and uniqueness of Nash equilibrium, under certain conditions, in various cases. Xu and Zhang [25] used a sample average approximation method to solve stochastic Nash equilibrium problems. Jadamba and Raciti [10] used a variational inequality approach on probabilistic Lebesgue spaces to study stochastic Nash games. The stochastic Nash games under expected payoff criterion using stochastic variational inequalities are considered in [11,15,16,26] and references therein. In these papers, the stochastic approximation based schemes to compute the Nash equilibria of stochastic Nash games have been given.

The expected payoff criterion is more appropriate for the cases where the decision makers are risk neutral. The risk averse stochastic Nash games arising from electricity market using risk measures as CVaR and variance are considered in [14,21] and [6] respectively. A risk averse payoff criterion based on chance constraint programming [3,20] has also received some attention in electricity market [7,17]. These games are called chance-constrained games. In [17], the randomness in payoffs is due to the installation of wind generators on the electricity market. The authors consider the case where the random variables that represent the amount of wind are independent normal random variables, and they also consider the case where the random vector follows a multivariate normal distribution. In [7], the consumers' random demand is assumed to be normally distributed. A game theoretic situation in electricity market where the action sets are finite is considered in [23]. Although the players' payoffs are deterministic in [23], the counterpart of the model, where the payoffs are random variables, in chance-constrained game setting can be considered. Only few theoretical results on zero sum chance-constrained games with finite action sets of the players are available in the literature so far [1,2,4,5,24].

In this paper, we focus on the games where the payoffs of the players are random variables with known probability distributions. The case where probability distributions of the random payoffs are not known completely is considered in [22]. The authors use distributionally robust approach to handle these games. To the

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best of our knowledge, there is no result on the existence of a Nash equilibrium for a chance-constrained game even when the action sets of all the players are finite. We consider an n-player strategic game where the action set of each player is finite and the payoff vector of each player is a random vector. We formulate this problem as a chance-constrained game by considering that the payoff of each player is defined using a chance constraint. We show the existence of a Nash equilibrium for a chance-constrained game in the following cases.

1. If all the components of the payoff vector of each player are independent normal/Cauchy random variables, there exists a mixed strategy Nash equilibrium for a chance-constrained game. As a special case, if only one component of the payoff vector of each player is a random variable and all other components are deterministic, the Nash equilibrium existence result can be extended to all the continuous probability distributions whose quantile functions exist.
2. If the payoff vector of each player follows a multivariate elliptically symmetric distribution, there always exists a mixed strategy Nash equilibrium for a chance-constrained game.

The structure of the rest of the paper is as follows: in Section 2 we give the definition of a chance-constrained game. Existence of a mixed strategy Nash equilibrium is then given in Section 3.

2. The model

We consider an n-player strategic game. Let $I = \{1, 2, \ldots, n\}$ be a set of all players. For each $i \in I$, let A_i be a finite action set of player i and its generic element is denoted by a_i. A vector $a = (a_1, a_2, \ldots, a_n)$ denotes an action profile of the game. Let $A = \times_{i=1}^{n} A_i$ be the set of all action profiles of the game. Denote, $A_{-i} = \times_{j=1;j\neq i}^{n} A_j$, and $a_{-i} \in A_{-i}$ is a vector of actions $a_j, j \neq i$. The action set A_i of player i is also called the set of pure strategies of player i. A mixed strategy of a player is represented by a probability distribution over his action set. For each $i \in I$, let X_i be the set of mixed strategies of player i, i.e., the set of all probability distributions over the action set A_i. A mixed strategy $\tau_i \in X_i$ is represented by $\tau_i = (\tau_i(a_i))_{a_i \in A_i}$, where $\tau_i(a_i) \geq 0$ is a probability with which player i chooses an action a_i and $\sum_{a_i \in A_i} \tau_i(a_i) = 1$. Let $X = \times_{i=1}^{n} X_i$ be the set of all mixed strategy profiles of the game and its element is denoted by $\tau = (\tau_i)_{i\in I}$. Denote $X_{-i} = \times_{j=1;j\neq i}^{n} X_j$, and $\tau_{-i} \in X_{-i}$ is a vector of mixed strategies $\tau_j, j \neq i$. We define (ν_i, τ_{-i}) to be a strategy profile where player i uses strategy ν_i and each player $j, j \neq i$, uses strategy τ_j. Let $\tilde{r}_i = \left(\tilde{r}_i(a)\right)_{a\in A}$ be a payoff vector of player i whose components are real numbers. Specifically, player i gets payoff $\tilde{r}_i(a) \in \mathbb{R}$ at action profile a. For such games, Nash [18] showed that there always exists a Nash equilibrium in mixed strategies.

In real life applications, the players' payoffs may be random due to uncertainty caused by various external factors. Therefore, we consider the case where the payoffs of each player are random variables and follow a certain distribution. We denote the random payoff vector by $\tilde{r}_i^w = \left(\tilde{r}_i^w(a)\right)_{a\in A}$, where w is an uncertainty parameter. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then, for each $i \in I$, we can think of the random payoff vector $\tilde{r}_i^w$ as a function $\tilde{r}_i^w : \Omega \to \mathbb{R}^{|A|}$ as follows $\tilde{r}_i^w(\omega) = \left(\tilde{r}_i^w(a,\omega)\right)_{a\in A} \in \mathbb{R}^{|A|}$; $|A|$ is the cardinality of set A. At action profile $a \in A$, $\tilde{r}_i^w(a,\omega)$ is the realized random payoff of player i. Then, for a given mixed strategy profile $\tau \in X$ and an $\omega \in \Omega$, the realized random payoff of player i, $i \in I$, is defined by,

$$r_i^w(\tau, \omega) = \sum_{a\in A} \prod_{j=1}^{n} \tau_j(a_j) \tilde{r}_i^w(a, \omega). \tag{2.1}$$

The random payoff $r_i^w(\tau)$ defined by (2.1) is a univariate random variable for all $\tau \in X$, and it is a linear combination of the components of the random payoff vector $(\tilde{r}_i^w(a))_{a\in A}$. We assume that each player uses satisficing payoff criterion where the payoff function of each player is defined using a chance constraint. At strategy profile $\tau \in X$, the payoff of each player is the highest level of his payoff that he can attain with at least a specified level of confidence. The confidence level of each player is given a priori and it is known to other players. Let $\alpha_i \in [0, 1]$ be the confidence level of player i and $\alpha = (\alpha_i)_{i\in I}$ be a confidence level vector. For a given strategy profile $\tau \in X$, and a given confidence level vector α the payoff function of player i, $i \in I$, is given by

$$u_i^{\alpha_i}(\tau) = \sup\left\{\gamma | P\left(\{\omega | r_i^w(\tau, \omega) \geq \gamma\}\right) \geq \alpha_i\right\}. \tag{2.2}$$

We assume that the probability distributions of the payoffs of each player are known to all the players. Then, for a given $\alpha \in [0, 1]^n$, the payoff function of a player defined by (2.2) is known to all the players. That is, for a given $\alpha \in [0, 1]^n$, the above chance-constrained game is a non-cooperative game with complete information. The set of best response strategies of player i, $i \in I$, against a given strategy profile τ_{-i} of other players is given by

$$BR_i^{\alpha_i}(\tau_{-i}) = \left\{\bar{\tau}_i \in X_i | u_i^{\alpha_i}(\bar{\tau}_i, \tau_{-i}) \geq u_i^{\alpha_i}(\tau_i, \tau_{-i}),\ \forall \tau_i \in X_i\right\}.$$

Next, we give the definition of Nash equilibrium.

Definition 2.1 (*Nash Equilibrium*). A strategy profile $\tau^* \in X$ is said to be a Nash equilibrium of a chance-constrained game for a given $\alpha \in [0, 1]^n$, if for all $i \in I$, the following inequality holds,

$$u_i^{\alpha_i}(\tau_i^*, \tau_{-i}^*) \geq u_i^{\alpha_i}(\tau_i, \tau_{-i}^*),\ \forall \tau_i \in X_i.$$

That is, τ^* is a Nash equilibrium if and only if $\tau_i^* \in BR_i^{\alpha_i}(\tau_{-i}^*)$ for all $i \in I$.

3. Existence of Nash equilibrium

We assume that the payoffs of each player are random variables following a certain distribution. We consider various cases and show the existence of a mixed strategy Nash equilibrium of chance-constrained game for different values of α.

3.1. Payoffs following normal/Cauchy distribution

For a given strategy profile $\tau \in X$, the probability distribution of random payoff $r_i^w(\tau)$, $i \in I$, plays an important role in defining the payoff function of player i given by (2.2). For each $i \in I$, $r_i^w(\tau)$ is a linear combination of the components of the random payoff vector $(\tilde{r}_i^w(a))_{a\in A}$. We consider the probability distributions, for the components of the random payoff vector $(\tilde{r}_i^w(a))_{a\in A}$, that are closed under the linear combination. It is well known that the independent random variables following normal or Cauchy distributions possess this property [12]. We first consider the case where for each $i \in I$, $\{\tilde{r}_i^w(a)\}_{a\in A}$ are independent normal random variables, where the mean and variance of $\tilde{r}_i^w(a)$ are $\mu_i(a)$ and $\sigma_i^2(a)$ respectively. Then, for $\tau \in X$, $r_i^w(\tau)$ follows a normal distribution with mean $\bar{\mu}_i(\tau) = \sum_{a\in A} \prod_{j=1}^{n} \tau_j(a_j)\mu_i(a)$ and variance $\bar{\sigma}_i^2(\tau) = \sum_{a\in A} \prod_{j=1}^{n} \tau_j^2(a_j)\sigma_i^2(a)$, and $Z_i^N = \frac{r_i^w(\tau)-\bar{\mu}_i(\tau)}{\bar{\sigma}_i(\tau)}$ follows a standard normal distribution. Let $F_{Z_i^N}^{-1}(\cdot)$, $i \in I$, be a quantile function of a standard normal distribution. From (2.2), for a given $\tau \in X$ and α, we have,

$$\begin{aligned} u_i^{\alpha_i}(\tau) &= \sup\left\{\gamma | P\left(\{\omega | r_i^w(\tau, \omega) \geq \gamma\}\right) \geq \alpha_i\right\} \\ &= \sup\left\{\gamma | P\left(Z_i^N \leq \frac{\gamma - \bar{\mu}_i(\tau)}{\bar{\sigma}_i(\tau)}\right) \leq 1 - \alpha_i\right\} \\ &= \bar{\mu}_i(\tau) + \bar{\sigma}_i(\tau) F_{Z_i^N}^{-1}(1 - \alpha_i). \end{aligned}$$