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Improved bounds on the probability of the union of events some of whose intersections are empty

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ABSTRACT

We formulate a linear program whose optimal objective function value can be used in other formulations to yield improved upper and lower bounds on the probability of the union of events if we know some empty intersections of small numbers of events. The LP relaxation of an extension of the maximum independent set problem provides an upper bound on the largest number of events that have a nonempty intersection. We present numerical experiments demonstrating the effectiveness of our formulation.

1. Introduction

Computing the probability of the union of events is important in reliability theory, stochastic programming, and other sciences concerned with stochastic systems. In network reliability, consider a communication network with nodes and links, each with a probability of failure. The two-terminal reliability of a pair of nodes is the probability of the union of events, each of which occurs when a path between the two nodes consists of links without failure. The all-terminal reliability is the probability of the union of events, each of which occurs when a spanning tree of the network consists of links without failure. In probabilistic constrained stochastic programming, a joint probabilistic constraint for random variables $\xi_1, \ldots, \xi_n$ specifies a lower bound on $\mathbb{P}(\xi_1 \le x_1 \cap \cdots \cap \xi_n \le x_n) = 1 - \mathbb{P}(\xi_1 > x_1 \cup \cdots \cup \xi_n > x_n)$, which involves the probability of the union of events. Although it is very hard to compute the exact probability of the union of a large number of events, we can compute its approximation by using the probabilities of individual events and intersections of a small number of events.

Let $\{A_1, \ldots, A_n\}$ be any set of n $(\in \mathbb{Z}_{>0})$ events in an arbitrary probability space and

$$N := \{1, \ldots, n\}$$

be the set of all positive integers at most n. For any finite set I, we designate its cardinality (i.e., the number of elements in I) by $|I|$.

For any subset $I \subseteq N$, we introduce a notation for the probability of the intersection of events A_i for all $i \in I$ as follows:

$$p_I := \mathbb{P}\left(\bigcap_{j\in I} A_j\right).$$

We introduce binomial moments S_i for all $i \in N$ as follows:

$$S_i := \sum_{I\subseteq N:\,|I|=i} p_I. \tag{1}$$

Let ν denote the random number of events among $A_1 \ldots, A_n$ that occur. Then we have the following relation for all $i \in N$:

$$\mathbb{E}\left[\binom{\nu}{i}\right] = \sum_{j=0}^{n} \binom{j}{i} \mathbb{P}(\nu = j) = S_i,$$

where we employ the extended definition of binomial coefficients, in which $\binom{j}{i} = 0$ for all $i, j \in \mathbb{Z}_{\ge 0}$ such that $i > j$. The value S_i is called the ith binomial moment of ν.

The classical inclusion–exclusion principle [10] (see also Prékopa [21]) gives the probability of the union of events by using binomial moments S_i for all $i \in N$ as follows:

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) = S_1 - S_2 + \cdots + (-1)^{n-1} S_n. \tag{2}$$

However, this formula is impractical if the number of events n is large, in which case the calculation of S_i is intractable unless i is small enough (close to 1) or large enough (close to n) since we need $\binom{n}{i}$ sums in (1). We can still calculate S_i for a few small i and compute lower and upper bounds on the probability. Let $m \in N$ be the largest number of events the probability of whose intersection

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is used in approximating the probability of the union and

$$M := \{1, \dots, m\}$$

be the set of all positive integers at most m; we use only p_I for all $I \subseteq N$ such that $|I| \in M$, from which $S_1, \dots, S_m$ can be calculated by (1). In practice, we usually consider a small $m \ll n$, mostly such as $m = 2, 3, 4$. The well-known Bonferroni inequalities (or bounds) [2] state that for any $m \in N$,

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) \begin{Bmatrix} \geq \\ \leq \end{Bmatrix} S_1 - S_2 + \cdots + (-1)^{m-1} S_m \quad \begin{cases} \text{if } m \text{ is even} \\ \text{if } m \text{ is odd.} \end{cases}$$

These bounds are usually very weak, often out of [0, 1]. Then the best possible (also called sharp) bounds using $S_1, \dots, S_m$ for a small m have been found. In the case $m = 2$, the sharp lower and upper bounds expressed as closed forms in terms of n, S_1, S_2 are obtained by Dawson and Sankoff [8] and Kwerel [15], respectively (see also Prékopa [18] and Boros and Prékopa [3]). In the case $m = 3$, the sharp bounds expressed as closed forms in terms of n, S_1, S_2, S_3 are obtained by Boros and Prékopa [3]. In the case $m = 4$, the sharp upper bound expressed as a closed form in terms of n, S_1, S_2, S_3, S_4 is obtained by Boros and Prékopa [3]. For a general m, Prékopa [18] observed that all these sharp bounds are optimal objective function values of certain linear programs, known as binomial moment problems. Furthermore, sharp bounds on the probabilities that exactly/at least r events occur are given in Prékopa [19], and the bounds on the probabilities and expectations of convex functions of discrete random variables are given in Prékopa [20]. A binomial moment S_i carries an aggregated information over event probabilities p_I, using every one of which without aggregation we can obtain better bounds. Hailperin [14] formulated linear programs with an exponential number of variables, known as Boolean probability bounding problems, which give sharp bounds using disaggregated event probabilities.

A pair of binomial moment problems (also called the aggregated LP problems, for probability bounds of the union of events) is formulated as follows [18]:

$$\begin{aligned} \text{min/max} \quad & \sum_{j \in N} x_j \\ \text{subject to} \quad & \sum_{j \in N} \binom{j}{i} x_j = S_i \quad \text{for } i \in M \\ & x_j \geq 0 \quad \text{for } j \in N. \end{aligned} \tag{3}$$

The optimal objective function values of these minimization and maximization problems are the sharp lower and upper bounds, respectively, on the probability of the union of n events that can be computed using only the aggregated information S_i for all $i \in M$.

A pair of Boolean probability bounding problems (also called the disaggregated LP problems, for probability bounds of the union of events) is formulated as follows [14]:

$$\begin{aligned} \text{min/max} \quad & \sum_{J \subseteq N:\, |J| \in N} x_J \\ \text{subject to} \quad & \sum_{J \subseteq N:\, |J| \in N} a_{IJ} x_J = p_I \quad \text{for } I \subseteq N : \ |I| \in M \\ & x_J \geq 0 \quad \text{for } J \subseteq N : \ |J| \in N, \end{aligned} \tag{4a}$$

where we define

$$a_{IJ} := \begin{cases} 1 & \text{if } I \subseteq J \\ 0 & \text{otherwise.} \end{cases} \tag{4b}$$

The optimal objective function values of these minimization and maximization problems are the sharp lower and upper bounds, respectively, on the probability of the union of n events that can be computed using only the disaggregated information p_I for all $I \subseteq N$ such that $|I| \in M$. Since such a set of event probabilities is almost always the most detailed information we can use, the bounds are the best possible we can expect in general. These problems are, however, impractical owing to an exponential number $(2^n - 1)$ of the decision variables x_J.

Although we cannot solve the disaggregated LP problems for a large number of events n in practice, we may extract useful information from disaggregated event probabilities without dealing with an exponential size and use it with their aggregations to obtain improved bounds.

2. Improved bounds

We assume that all probabilities of intersections of small numbers of events (formally, p_I for all $I \subseteq N$ such that $|I| \in M \subset N$) are known and some of them are 0 (i.e., such intersections are empty). The smallest number of events that have an empty intersection is 2 since we may use only nonempty individual events when computing their union; hence, we assume that $m \geq 2$. For any nonempty subsets $I, J \subseteq N$ such that $I \subseteq J$ and any $\varepsilon \in [0, 1]$, we have the implication $p_I \leq \varepsilon \Longrightarrow p_J \leq \varepsilon$. In particular when $\varepsilon = 0$, we have the following implication:

$$p_I = 0 \Longrightarrow p_J = 0,$$

which suggests that we are likely to have more empty intersections of a larger number of events even if we have a few empty intersections of small numbers of events; if at least one intersection is empty, then the intersection of the largest number of events is empty (i.e., $p_N = 0$, where $|N| = n$). Therefore, there exists the largest number $r \in \{1, \dots, n-1\}$ of events that have a nonempty intersection; all intersections of a larger number of events are empty (i.e., $p_I = 0$ for all I such that $|I| \in \{r+1, \dots, n\}$). We want to find r, but since it is impossible to find the exact number with known limited information about event probabilities, we find an upper bound on it instead.

First, we consider the case $m = 2$ for ease of understanding. We can find the sharp upper bound by solving the maximum independent set problem as follows. Consider a simple graph $G = (V, E)$ of $n = |V|$ vertices corresponding to n events $A_1, \dots, A_n$, respectively. The graph G has an edge between two distinct vertices $i, j \in V$ if and only if the intersection of the corresponding two events is empty: $(i, j) \in E \iff \mathbb{P}(A_i \cap A_j) = 0$. An independent set (also known as a stable set) is a subset of V that has no edge between its vertices. The size of an independent set is the number of vertices in it. The maximum independent set problem is to find a largest independent set for a given undirected graph. Thus, the intersection of the events corresponding to the vertices in any larger subset than a maximum independent set is empty since the subset must contain a pair of connected vertices. The intersection of the events corresponding to the vertices in a maximum independent set could be empty or nonempty although all pairwise intersections of events are nonempty. Therefore, the size of a maximum independent set is the sharp upper bound knowing only all probabilities of individual and pairwise intersections of events. The problem is $\mathcal{NP}$-hard, so it is unlikely that we can find the exact solution efficiently. Instead, we can solve the following LP relaxation efficiently:

$$\begin{aligned} \text{maximize} \quad & \sum_{j \in V} x_j \\ \text{subject to} \quad & x_i + x_j \leq 1 \quad \text{for } (i, j) \in E, \\ & 0 \leq x_j \leq 1 \quad \text{for } j \in V. \end{aligned}$$

Now, we consider the general case $m \geq 2$, forgetting about the graph representation but thinking of extending the mathematical