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# A dynamic process to the core for multi-choice games

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ABSTRACT

In the framework of multi-choice games, we propose a dynamic process leading to the core which was introduced by van den Nouweland et al. (1995). Also, we prove that it converges if and only if the core is nonempty.



## 1. Introduction

The core of TU-games (games with transferable utility) was introduced by Gillies [5] in 1953. Subsequently, in 1998, Cesco [3] provided a transfer scheme leading to the nonempty core of a TU-game. Also he proved that such a transfer scheme converges if and only if the core of a TU-game is nonempty. In the literature, the core of multi-choice games was first introduced by van den Nouweland et al. [11]. This raises the question whether the result of Cesco [3] extends to multi-choice games. The answer is positive. The aim of this note is to describe a model of dynamic bargaining that converges to the core for multi-choice games. Also, such a transfer scheme converges if and only if the core of a multi-choice game is nonempty. In the framework of TU games, related results may be found in Stearns [10], Justman [6], Maschler and Owen [7], and so on. Stearns [10] described two transfer scheme that converges to the kernel [4] and the bargaining set [1], respectively. Justman [6] provided a transfer scheme that converges to the nucleolus [8]. Maschler and Owen [7] described a dynamic process that converges to the Shapley value.

As Cesco [3] pointed out that Bondareva [2] and Shapley [9] used the duality theorem in linear programming to prove that the core of a TU-game is nonempty if and only if the game is balanced. But, Cesco did not use the linear programming method. Cesco proved that a transfer scheme converges if and only if the core of a TU-game is nonempty. Similarly, van den Nouweland et al. [11] used the duality theorem to prove that the core of a multi-choice game is nonempty if and only if the game is balanced. But our proof does not use it.

The paper is organized as follows. In Section 2, we introduce the definitions and some notations. In Section 3, we present a transfer scheme and prove the convergence result. Also, we provide an algorithm for our convergence result. In Section 4, we provide a solid discussion.

## 2. Definitions and notations

Let $N = \{1, 2, \dots, n\}$ be a set of players and for $i \in N$, let $M_i = \{0, 1, 2, \dots, m_i\}$ be an action set of player $i$ which means player $i$ has $m_i$ activity levels at which he or she can play, where the action 0 means a player does not participate, here we denote $M_i \setminus \{0\}$ as $M_i^+$. The set of action vectors is denoted by $\prod_{i\in N} M_i = \{(\rho_1, \dots, \rho_n) : \rho_i \in M_i \text{ for all } i \in N\}$, where $\rho = (\rho_1, \dots, \rho_n)$ is called an action vector of $N$.

For convenience, we assume $m_i = m$ for all $i \in N$. A multi-choice game is a triple $(N, m, v)$, where $N$ is a set of players, $m \in \mathbb{N}$ is the number of activity levels for all players, and $v : \prod_{i\in N} M_i \to \mathbb{R}$ is the characteristic function satisfying $v(\theta) = 0$, where $\theta$ denotes the zero vector in $\mathbb{R}^N$. For $S \subseteq N$, let $e^S$ be the vector in $\mathbb{R}^N$ satisfying $e_i^S = 0$ if $i \notin S$ and $e_i^S = 1$ if $i \in S$. In this note, without loss of generality, we assume that $v$ is zero–one normalized, i.e., $v(me^N) = 1$ and $v(je^{\{i\}}) = 0$ for all $i \in N$ and $j \in M_i^+$. And if there is no confusion we will denote $(N, m, v)$ by $v$.

Let $v$ be a game, we denote $L(v) = \{(i, j) : i \in N, j \in M_i^+\}$, and $x = (x_{ij})_{i\in N, j\in M_i^+} \in \mathbb{R}^{L(v)}$ be a payoff configuration, where $x_{ij}$ denotes the increase in payoff to player $i$ corresponding to a change of activity from level $j-1$ to level $j$ by this player, following van den Nouweland et al. [11]. For any $\rho = (\rho_1, \dots, \rho_n) \in \prod_{i\in N} M_i$,

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denote $S_\rho = \{i \in N : \rho_i \neq 0\}$, $L_{S_\rho}(v) = \{(i,j) : i \in S_\rho, 1 \leq j \leq \rho_i\}$ and $x(\rho) = \sum_{i\in S_\rho}\sum_{j=1}^{\rho_i} x_{ij}$. For $F \subset L(v)$ with $F \neq \emptyset$ and $F \neq L(v)$, let $\beta_F$ be the payoff configuration in $\mathbb{R}^{L(v)}$ satisfying $(\beta_F)_{ij} = \frac{1}{|F|}$ if $(i,j) \in F$ and $(\beta_F)_{ij} = \frac{-1}{|F^c|}$ if $(i,j) \notin F$, i.e.

$$\beta_F = \frac{1_F}{|F|} - \frac{1_{F^c}}{|F^c|},$$

where $F^c$ denotes the complement of $F$ in $L(v)$, $|F|$ and $|F^c|$ denote the cardinality of $F$ and $F^c$, respectively. $1_F$ is the payoff configuration in $\mathbb{R}^{L(v)}$ defined by $(1_F)_{ij} = 1$ if $(i,j) \in F$ and 0, otherwise.

**Remark 2.1.** Since $F$ is finite, it follows that there are two positive constants $k$ and $K$ such that $k \leq \|\beta_F\|_2 \leq K$.

Let $v$ be a game. The pre-imputation set of $v$ is defined by

$$E(v) = \left\{x = (x_{ij})_{i\in N, j\in M_i^+} : x(me^N) = \sum_{i\in N}\sum_{j=1}^{m} x_{ij} = 1\right\}.$$

Let $v$ be a game, $\rho = (\rho_1, \ldots, \rho_n) \in \prod_{i\in N} M_i$, and $x \in E(v)$. We define the excess function of $\rho$ in $x$ for $v$ by

$$e(\rho, x, v) = v(\rho) - x(\rho).$$

The core of multi-choice games is defined by van den Nouweland et al. [11] as follows.

**Definition 2.2.** Let $v$ be a game. The core $C(v)$ of the game $v$ consists of all $x \in E(v)$ that satisfy $e(\rho, x, v) \leq 0$ for all $\rho \in \prod_{i\in N} M_i$, i.e.

$$C(v) = \left\{x \in E(v) : e(\rho, x, v) \leq 0 \quad \text{for all } \rho \in \prod_{i\in N} M_i\right\}.$$

## 3. Transfer scheme and convergence result

In this section, we present a transfer scheme in the setting of multi-choice games, and prove that the maximal transfer scheme converges if and only if the core is nonempty. First, we present a transfer and a transfer scheme in the setting of multi-choice games as follows.

Let $v$ be a game, $x$ be an arbitrary payoff configuration in $E(v)$, $\rho$ be an action vector in $\prod_{i\in N} M_i \setminus \{\theta, me^N\}$ and $e(\rho, x, v) \geq 0$, then we use the following terminologies:

1. $y$ is said to result from $x$ by a transfer of size $e(\rho, x, v)$ from $[L_{S_\rho}(v)]^c$ to $L_{S_\rho}(v)$ if
$$\begin{aligned} y &= x + e(\rho, x, v)\beta_{L_{S_\rho}(v)} \\ &= x + e(\rho, x, v)\left(\frac{1_{L_{S_\rho}(v)}}{|L_{S_\rho}(v)|} - \frac{1_{[L_{S_\rho}(v)]^c}}{|[L_{S_\rho}(v)]^c|}\right). \end{aligned}$$
2. A transfer scheme is a sequence $\{x^r\}_{r=1}^{\infty}$ with $x^{r+1}$ is resulted from $x^r$ by a transfer of size $e(\rho^r, x^r, v)$ from $[L_{S_{\rho^r}}(v)]^c$ to $L_{S_{\rho^r}}(v)$, i.e.,
$$\begin{aligned} x^{r+1} &= x^r + e(\rho^r, x^r, v)\beta_{L_{S_{\rho^r}}(v)} \\ &= x^r + e(\rho^r, x^r, v)\left(\frac{1_{L_{S_{\rho^r}}(v)}}{|L_{S_{\rho^r}}(v)|} - \frac{1_{[L_{S_{\rho^r}}(v)]^c}}{|[L_{S_{\rho^r}}(v)]^c|}\right) \end{aligned}$$
for all $r \geq 1$.
3. A transfer scheme $\{x^r\}_{r=1}^{\infty}$ is said to be maximal if
$$e(\rho^r, x^r, v) = \max\{e(\rho, x^r, v) : e(\rho, x^r, v) \geq 0 \text{ for all } \rho\}$$
for all $r \geq 1$.

In fact, a payoff configuration $x = (x_{ij})_{i\in N, j\in M_i^+}$ in $\mathbb{R}^{L(v)}$ is a matrix in $\mathbb{R}^{n\times m}$ exactly. For convenience, in the following proofs, we will regard a payoff configuration $x$ as a vector

$$(x_{11}, x_{12}, \ldots, x_{1m}, x_{21}, x_{22}, \ldots, x_{2m}, x_{31}, \ldots, x_{3m}, x_{41}, \ldots, x_{4m}, \ldots, x_{n1}, \ldots, x_{nm}).$$

The inner product of two vectors $x$ and $y$ is denoted by $\langle x, y\rangle$, and two vectors $x$ and $y$ are orthogonal if $\langle x, y\rangle = 0$.

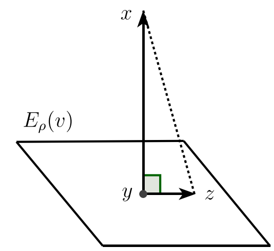


**Fig. 1.** A right triangle.

**Lemma 3.1.** *Let $v$ be a game and $\rho \in \prod_{i\in N} M_i \setminus \{\theta, me^N\}$. Then $\beta_{L_{S_\rho}(v)}$ is orthogonal to the affine submanifold*

$$E_\rho(v) = \left\{x \in E(v) : e(\rho, x, v) = 0\right\}.$$

**Proof.** Let $x, y \in E_\rho(v)$. Then

$$\langle \beta_{L_{S_\rho}(v)}, x - y\rangle = \frac{\sum_{i\in S_\rho}\sum_{j=1}^{\rho_i} x_{ij} - \sum_{i\in S_\rho}\sum_{j=1}^{\rho_i} y_{ij}}{|L_{S_\rho}(v)|} - \frac{\sum_{i\in S_\rho}\sum_{j=\rho_i+1}^{m} x_{ij} - \sum_{i\in S_\rho}\sum_{j=\rho_i+1}^{m} y_{ij}}{|[L_{S_\rho}(v)]^c|}.$$

Since $x, y \in E_\rho(v)$, by $e(\rho, x, v) = e(\rho, y, v) = 0$ and $x, y \in E(v)$, we have

$$\sum_{i\in S_\rho}\sum_{j=1}^{\rho_i} x_{ij} = \sum_{i\in S_\rho}\sum_{j=1}^{\rho_i} y_{ij} = v(\rho) \quad \text{and}$$

$$\sum_{i\in S_\rho}\sum_{j=\rho_i+1}^{m} x_{ij} = \sum_{i\in S_\rho}\sum_{j=\rho_i+1}^{m} y_{ij} = 1 - v(\rho), \quad \text{respectively.}$$

Hence $\langle \beta_{L_{S_\rho}(v)}, x - y\rangle = 0$. □

**Lemma 3.2.** *Let $v$ be a game with nonempty core $C(v)$. Let $x \in E(v)$ and $\rho$ be an action vector with $e(\rho, x, v) > 0$. If $y = x + e(\rho, x, v)\beta_{L_{S_\rho}(v)}$, then $\|y - z\|_2 < \|x - z\|_2$ for all $z \in C(v)$.*

**Proof.** Clearly, $y \in E_\rho(v)$. Let $z \in C(v)$, then $e(\rho, z, v) \leq 0$. Since $e(\rho, x, v) > 0$ and $e(\rho, z, v) \leq 0$, by definition of $E_\rho(v)$ (as in Lemma 3.1), $E_\rho(v)$ is a separating linear submanifold for $x$ and $z$. By $y = x + e(\rho, x, v)\beta_{L_{S_\rho}(v)}$, then $y - x = e(\rho, x, v)\beta_{L_{S_\rho}(v)}$, hence $y - x$ is orthogonal to $E_\rho(v)$ by Lemma 3.1. Next, we will show $\|y - z\|_2 < \|x - z\|_2$ by the property of "longer side has larger opposite angle". Two cases could be distinguished:

**case (1)**: when $z \in E_\rho(v)$:

Since $y - x$ is orthogonal to $E_\rho(v)$, this implies that $x, y, z$ form a triangle with right angle at $y$ (see Fig. 1). Hence $\|y - z\|_2 < \|x - z\|_2$.

**case (2)**: when $z \notin E_\rho(v)$:

In this case, $x, y, z$ form a line (see Fig. 2) or a triangle with obtuse angle at $y$ (see Fig. 3). Hence, $\|y - z\|_2 < \|x - z\|_2$. □

The following theorem shows the part of "only if" in our main result.

**Theorem 3.3.** *Let $v$ be a game. If every maximal transfer scheme converges to a point $x \in \mathbb{R}^{n\times m}$, then $x$ belongs to the core $C(v)$, i.e., the core $C(v)$ is nonempty.*