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# The Shapley value for directed graph games

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ABSTRACT

The Shapley value for directed graph (digraph) TU games with limited cooperation induced by a digraph prescribing the dominance relation among the players is introduced. It is defined as the average of the marginal contribution vectors corresponding to all permutations which do not violate the induced subordination of players. We study properties of this solution and its core stability. For digraph games with the digraphs being directed cycles an axiomatization of the solution is obtained.


## 1. Introduction

In classical cooperative game theory it is assumed that any coalition of players may form and is able to obtain payoffs for its members. Problem is how much payoff each player should receive. However, in many practical situations the set of feasible coalitions is limited by some social, economical, hierarchical, or technical structure. One of the most famous singleton solutions for cooperative games with transferable utility (TU games), where payoffs can be distributed freely among the players, is the Shapley value [8] defined as the average of the marginal contribution vectors corresponding to all permutations on the players. Several adaptations of the Shapley value for models of games with limited cooperation among the players are well known in the literature, cf. Aumann and Drèze [1] and Owen [7] for games with coalition structure, Myerson [6] for games with cooperation structure introduced by means of undirected graphs in which only the connected players are able to cooperate. For games with limited cooperation that is described in terms of (cycle-free) directed graphs (digraphs) we mention Gilles and Owen [3] for games with permission structure using the disjunctive approach and Gilles et al. [4] for such games using the conjunctive approach, and Faigle and Kern [2] for games with precedence constraints.

In this paper we assume that restricted cooperation is determined by an arbitrary digraph on the player set, the directed links of which prescribe the subordination among the players. For example, consider a society consisting of individuals with different opinions, possibly incomplete preferences, about the importance of several proposals or tasks that need to be completed. If the preferences of the individuals are aggregated by using majority voting, then it is well known that the resulting structure will be a directed graph on the set of alternatives. In this directed graph, a directed link from one proposal to another proposal means that the majority of the society thinks that the former one is more important than the latter one. If it is assumed that at each moment only one proposal or task can be performed, then when one is completed, the next one to be performed can be any of its immediate successors in the digraph or one of those the performance of which does not depend on it. In this example the digraph might not be cycle-free because directed cycles may stand for the well known Condorcet paradox.

On the class of digraph games, which are games with restricted cooperation determined by a digraph prescribing the dominance relation on the set of players, we introduce the so-called Shapley value for digraph games as the average of marginal contribution vectors corresponding to all permutations not violating the subordination of players. Contrary to the Myerson model, the feasible coalitions are not necessarily connected. We show that the Shapley value for digraph games meets efficiency, linearity, the restricted

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null player property, the restricted equal treatment property, is independent of inessential links, and is stable with respect to the appropriate core concept under a convexity type condition which is weaker than the usual convexity guaranteeing the core stability of the classical Shapley value. On the subclass of cycle digraph games for which the digraphs are directed cycles an axiomatization is provided.

Since precedence constraints are determined by a partial ordering on the player set which can be represented by a cycle-free digraph, the games under precedence constraints form a subclass of cycle-free digraph games on which the Shapley value for digraph games coincides with the Shapley value for games under precedence constraints of Faigle and Kern [2]. There is no straightforward relation of permission values for games with permission structure with the newly introduced Shapley value for digraph games. In games with permission structure players need permission from their predecessors in order to cooperate, at least one of them for disjunctive approach and all of them for conjunctive approach. In both cases a permission-restricted TU game is derived from the given TU game taking into account the permission structure and the disjunctive and conjunctive permission values for games with permission structure are defined as the Shapley value of the corresponding permission-restricted games.

The structure of the paper is as follows. Section 2 contains preliminaries. Section 3 introduces the Shapley value for digraph games and discusses its properties and stability. An axiomatization on the subclass of cycle digraph games is obtained in Section 4.

## 2. Preliminaries

A *cooperative game with transferable utility* (*TU game*) is a pair $(N, v)$, where $N = \{1, \dots, n\}$ is a finite set of $n \geq 2$ players and $v: 2^N \to \mathbb{R}$ is a *characteristic function* with $v(\emptyset) = 0$, assigning to any *coalition* $S \subseteq N$ its *worth* $v(S)$. The set of TU games with fixed player set $N$ is denoted $\mathcal{G}_N$. For simplicity of notation and if no ambiguity appears we write $v$ when we refer to a game $(N, v)$. It is well known (cf. Shapley [8]) that *unanimity games* $\{u_T\}_{\substack{T\subseteq N\\ T\neq\emptyset}}$, defined as $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise, form a basis in $\mathcal{G}_N$. A *value* on $\mathcal{G} \subseteq \mathcal{G}_N$ is a function $\xi: \mathcal{G} \to \mathbb{R}^N$ that assigns to every $v \in \mathcal{G}$ a vector $\xi(v) \in \mathbb{R}^N$ where $\xi_i(v)$ is the *payoff* to $i \in N$ in $v$. The *marginal contribution* of $i \in N$ to $S \subseteq N \setminus \{i\}$ in $v \in \mathcal{G}_N$ is given by $m_i^v(S) = v(S \cup \{i\}) - v(S)$. In the sequel we use standard notation $x(S) = \sum_{i\in S} x_i$ for any $x \in \mathbb{R}^N$ and $S \subseteq N$.

For a permutation $\pi: N \to N$, $\pi(i)$ is the position of player $i \in N$ in $\pi$, $P_\pi(i) = \{j \in N \mid \pi(j) < \pi(i)\}$ is the set of predecessors of $i$ in $\pi$, and $\bar{P}_\pi(i) = P_\pi(i) \cup \{i\}$. In what follows we identify a permutation $\pi$ with the vector $(\pi(1), \dots, \pi(n))$. Let $\Pi$ be the set of permutations on $N$. For $v \in \mathcal{G}_N$ and $\pi \in \Pi$ the *marginal contribution vector* $\bar{m}^v(\pi) \in \mathbb{R}^N$ is given by $\bar{m}_i^v(\pi) = m_i^v(P_\pi(i)) = v(\bar{P}_\pi(i)) - v(P_\pi(i))$ for all $i \in N$. The *Shapley value* of $v \in \mathcal{G}^N$ is given by $Sh(v) = \sum_{\pi\in\Pi} \bar{m}^v(\pi)/n!$.

A *graph* on $N$ consists of $N$ as the set of nodes and for a *directed graph* (*digraph*) a collection of ordered pairs $\Gamma \subseteq \{(i,j) \mid i,j \in N,\ i \neq j\}$ as the set of *directed links* (*arcs*) from one player to another in $N$, and for an *undirected graph* a collection of unordered pairs $\Gamma \subseteq \{\{i,j\} \mid i,j \in N,\ i \neq j\}$ as the set of *links* (*edges*) between two players in $N$. Observe that an undirected graph can be considered as a digraph for which $(i,j) \in \Gamma$ iff $(j,i) \in \Gamma$. We say that a digraph $\Gamma$ contains an undirected link $\{i,j\}$ and write $\{i,j\} \in \Gamma$ if $(i,j), (j,i) \in \Gamma$. The set of digraphs on fixed $N$ we denote $\Gamma_N$. For $\Gamma \in \Gamma_N$ and $S \subseteq N$, $\Gamma|_S = \{(i,j) \in \Gamma \mid i,j \in S\}$ is the *subgraph of* $\Gamma$ on $S$. Given $\Gamma \in \Gamma_N$ a sequence of different players $(i_1, \dots, i_r)$, $r \geq 2$, is a *path* in $\Gamma$ between $i_1$ and $i_r$ if $\{(i_h, i_{h+1}), (i_{h+1}, i_h)\} \cap \Gamma \neq \emptyset$ for $h = 1, \dots, r-1$, and a *directed path* in $\Gamma$ from $i_1$ to $i_r$ if $(i_h, i_{h+1}) \in \Gamma$ for $h = 1, \dots, r-1$. A directed path $(i_1, \dots, i_r)$ is a *directed cycle* if $(i_r, i_1) \in \Gamma$ and when $r \geq 3$, both the path does not contain undirected links and $(i_1, i_r) \notin \Gamma$. $\Gamma$ is *cycle-free* if it contains no directed cycles. Players $i, j \in N$ are *connected* in $\Gamma$ if there exists a path in $\Gamma$ between $i$ and $j$. $\Gamma$ is *connected* if any $i, j \in N$, $i \neq j$, are connected in $\Gamma$. $S \subseteq N$ is *connected* in $\Gamma$ if $\Gamma|_S$ is connected. For $S \subseteq N$, $C^\Gamma(S)$ denotes the collection of subsets of $S$ connected in $\Gamma$, $S/\Gamma$ is the collection of maximal connected subsets, called *components*, of $S$ in $\Gamma$. For $i, j \in N$ if there exists a directed path in $\Gamma$ from $i$ to $j$, then $j$ is a *successor* of $i$ and $i$ is a *predecessor* of $j$ in $\Gamma$. If $(i,j) \in \Gamma$, then $j$ is an *immediate successor* of $i$ and $i$ is an *immediate predecessor* of $j$ in $\Gamma$. For $i \in N$, $S^\Gamma(i)$ denotes the set of successors of $i$ in $\Gamma$ and $\bar{S}^\Gamma(i) = S^\Gamma(i) \cup \{i\}$. A *chain* on $N$ is a connected cycle-free digraph on $N$ in which each player has at most one immediate successor and one immediate predecessor.

For $\Gamma \in \Gamma_N$, $S \subseteq N$ and $i, j \in S$, $i$ *dominates* $j$ in $\Gamma|_S$, denoted $i \succ_{\Gamma|_S} j$, if $j \in S^{\Gamma|_S}(i)$ and $i \notin S^{\Gamma|_S}(j)$. Observe that the dominance relation between two players may differ between different coalitions they both belong to. Player $i \in S$ is *undominated* in $\Gamma|_S$ if no player in $S$ dominates $i$ in $\Gamma|_S$, i.e., $i \in S^{\Gamma|_S}(j)$ implies $j \in S^{\Gamma|_S}(i)$. Note that a player undominated in $\Gamma|_S$ either has no predecessor in $\Gamma|_S$ or lies on a directed cycle in $\Gamma|_S$. $U^\Gamma(S)$ denotes the set of players undominated in $\Gamma|_S$. Since $N$ is finite, $U^\Gamma(S) \neq \emptyset$ for $\emptyset \neq S \subseteq N$.

A pair $(v, \Gamma)$ of $v \in \mathcal{G}_N$ and $\Gamma \in \Gamma_N$ constitutes a *directed graph game*, or a *digraph game*. The set of digraph games on fixed $N$ is denoted $\mathcal{G}_N^\Gamma$. A *value* on $\mathcal{G} \subseteq \mathcal{G}_N^\Gamma$ is a function $\xi: \mathcal{G} \to \mathbb{R}^N$ assigning to every $(v, \Gamma) \in \mathcal{G}$ a payoff vector $\xi(v, \Gamma)$.

## 3. The Shapley value for digraph games

In a digraph game the digraph prescribes a dominance relation between the players that puts restrictions on the feasibility of coalitions. Assuming that in order to cooperate players may join only the players not dominating them, the set of feasible coalitions of a digraph game consists of hierarchical coalitions.

Given $\Gamma \in \Gamma_N$, $S \subseteq N$ is a *hierarchical coalition* in $\Gamma$ if $i \in S$, $(i,j) \in \Gamma$, and $i \notin S^\Gamma(j)$ imply $\bar{S}^\Gamma(j) \subset S$.

If a player in a hierarchical coalition dominates an immediate successor, then the coalition also contains this latter player and all his successors. Every hierarchical coalition preserves the subordination of players and therefore is feasible. For a cycle-free $\Gamma \in \Gamma_N$, $S \subseteq N$ is hierarchical iff every successor of any $i \in S$ in $\Gamma$ belongs to $S$, i.e., $\bar{S}^\Gamma(i) \subseteq S$ for all $i \in S$. So, for a cycle-free digraph the set of hierarchical coalitions coincides with the set of feasible coalitions in Faigle and Kern [2] when the precedence constraints are induced by the same digraph. Note that both the empty and grand coalitions are hierarchical. A hierarchical coalition is not necessarily connected. In an undirected graph, in particular in the empty graph, every coalition is hierarchical. For $\Gamma \in \Gamma_N$, $H(\Gamma)$ denotes the set of coalitions hierarchical in $\Gamma$ and $H^c(\Gamma)$ its subset of all connected coalitions. Observe that $S, T \in H(\Gamma)$ implies $S \cup T$, $S \cap T \in H(\Gamma)$.

Given $\Gamma \in \Gamma_N$, $\pi \in \Pi$ is *consistent with* $\Gamma$ if it preserves the subordination of players determined by $\Gamma$, i.e., $\pi(j) < \pi(i)$ only if $j \not\succ_{\Gamma|_{\bar{P}_\pi(i)}} i$.

For $\Gamma \in \Gamma_N$, $\Pi^\Gamma$ denotes the set of permutations consistent with $\Gamma$. Since $N$ is finite, $\Pi^\Gamma \neq \emptyset$.

**Remark 3.1.** For every $\pi \in \Pi^\Gamma$ each player is undominated in the subgraph of $\Gamma$ on the set composed by this player and his predecessors in $\pi$, i.e., $i \in U^\Gamma(\bar{P}_\pi(i))$ for all $i \in N$.

The next proposition shows that every consistent permutation generates a sequence of feasible coalitions consisting of a player and his predecessors in the permutation.