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Extended formulations for vertex cover

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ABSTRACT

The vertex cover polytopes of graphs do not admit polynomial-size extended formulations. This motivates the search for polyhedral analogues to approximation algorithms and fixed-parameter tractable (FPT) algorithms. In this paper, we take the FPT approach and study the k-vertex cover polytope (the convex hull of vertex covers of size k). Our main result is that there are extended formulations of size $O(1.47^k + kn)$. We also provide FPT extended formulations for solutions of size k to instances of d-HITTING SET.

1. Introduction

Two of the most well-studied problems in combinatorial optimization are the minimum vertex cover problem and the maximum independent set problem. Their polyhedral representations, the vertex cover polytope VC(G) and the independent (or stable) set polytope STAB(G), have been the subject of numerous studies [27,25,26,31], and can be defined as follows. For a graph $G = (V, E)$,

$$\text{VC}(G) := \text{conv.hull}\left\{x^S \mid S \subseteq V \text{ is a vertex cover for } G\right\}$$

$$\text{STAB}(G) := \text{conv.hull}\left\{x^S \mid S \subseteq V \text{ is an independent set in } G\right\}$$

where x^S denotes the characteristic vector of S. These two polytopes are closely related to each other. Indeed, letting $\mathbf{1}$ be a vector of ones, it is easy to see that

$$\text{VC}(G) = \{x \mid (\mathbf{1} - x) \in \text{STAB}(G)\}.$$

One reason for studying these polytopes is that exact, partial, or even approximate descriptions of them by linear inequalities can be used to help solve integer programs with set packing constraints [27].

Due to the difficulty of the minimum vertex cover problem and the maximum independent set problem, VC(G) and STAB(G) are likely to be complicated. Indeed, they can have exponentially many facets even when G is series-parallel, but, in this case, VC(G) and STAB(G) admit linear-size *extended formulations* [3]. (The size of an extended formulation is the number of linear inequalities in its description.)

However, small extended formulations for VC(G) and STAB(G) are the exception rather than the rule. For some classes of graphs, $2^{\Omega(\sqrt{n})}$ linear inequalities are needed [16], and a lower bound of $2^{\Omega(n)}$ has been conjectured [7]. This motivates the search for approximate or fixed-parameter tractable (FPT) extended formulations. The polyhedral approximability of VC(G) and STAB(G) was recently studied by Bazzi et al. [4]. In this paper, we take the FPT approach. An extended formulation is said to be FPT if its size is bounded above by $f(k)n^{O(1)}$, where k is the parameter of choice, f is a function that depends only on k, and n is the input size.

To formalize our approach, consider the k-vertex cover polytope VC$_k$(G) of a graph G, which is the convex hull of vertex covers of G that have size k.

$$\text{VC}_k(G) := \text{conv.hull}\left\{x^S \mid S \subseteq V \text{ is a vertex cover for } G \text{ and } |S| = k\right\}.$$

The problem of determining whether a graph has a vertex cover of size k, i.e., the k-vertex cover problem, is the prototypical problem in the FPT literature, so there are numerous structural results that can be used when developing extended formulations. For example, we use Damaschke's [15] refinement of Sam Buss's kernel for k-vertex cover [11] and decomposition ideas from Chen et al. [12].

1.1. Our contributions

Our main result is that k-vertex cover polytopes of n-vertex graphs admit extended formulations of size $O(1.47^k + kn)$. En route

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to proving this, we exhibit size $O(n)$ formulations for the case of graphs of maximum degree at most two. In its proof, we use known results regarding the cardinality-constrained matching polytope and t-perfect graphs.

Before showing this, we provide simple extended formulations for the hypergraph case. They are of size $O(d^k n)$ for d-uniform hypergraphs. (In a d-uniform hypergraph, each hyperedge has d vertices.) This is equivalent to solutions of size k to instances of d-Hitting Set. Our results hold when each hyperedge has *at most* d vertices, but we state everything for d-uniform hypergraphs for sake of exposition. This implies size $O(2^k n)$ extended formulations for the standard vertex cover problem. This bound is worse than what we eventually will show, but it is easier to obtain.

1.2. Preliminaries and related work

Definition 1. Let $P = \{x \mid Ax \le b\} \subseteq \mathbb{R}^n$ be a polyhedron. A polyhedron $Q \subseteq \mathbb{R}^d$ is said to be an *extension* for P if $\text{proj}_x(Q) = P$, where $\text{proj}_x(Q) := \{x \mid \exists y : (x, y) \in Q(G)\}$. The *size* of an extension is the number of its facets.

A particular representation of an extension by linear inequalities is called an extended formulation. Consult the surveys of Conforti et al. [14] and Kaibel [19] for some notable extended formulations.

Definition 2. The *extension complexity* of a polyhedron P is

$\text{xc}(P) := \min\{\text{size}(Q) \mid Q \text{ is an extension for } P\}$.

We will use Balas' extended formulation for the union of polyhedra.

Theorem 1 (*Balas [1,2]*). *Consider q polytopes $P^i \subseteq \mathbb{R}^n$, $i = 1, \dots, q$ and write $P := \text{conv.hull}\left(\bigcup_{i=1}^q P^i\right)$. Then, $\text{xc}(P) \le q + \sum_{i=1}^q \text{xc}(P^i)$.*

Since the work of Yannakakis [34], there have been numerous advances showing that certain polytopes have high extension complexity. For example, polytopes associated with NP-hard problems such as the traveling salesman problem and the 0-1 knapsack problem admit no polynomial-size extended formulation [16,28]—irrespective of whether P = NP. There are even polynomial-time solvable problems such as matching that admit no polynomial-size extended formulations [29]. While the matching polytope does not admit a polyhedral equivalent to an FPTAS [8], it does admit FPT extended formulations parameterized by the number of edges in the matching [20].

In a draft of this paper, we asked whether there exist FPT extended formulations for independent set (parameterized by solution size). Gajarský et al. [17] answer this in the negative for arbitrary graphs, but provide FPT extended formulations for graphs of bounded expansion.

The independent set and vertex cover problems have both been studied from the perspective of approximate extended formulations. Independent set admits no polynomial-size *uniform* extended formulation that achieves an $O(n^{1-\epsilon})$ approximation for any constant $\epsilon > 0$ [9], which matches the inapproximability of the maximum independent set problem [18,35]. If we allow for *non-uniform* extended formulations, that is, the inequalities defining the feasible region need not be the same for every n-vertex graph, then $O(1)$-approximate formulations still require superpolynomial size [4]. Somewhat surprisingly, $O(n^{1/2})$-approximate extended formulations of size $O(n)$ exist, but they are NP-hard to construct [4]. For the vertex cover problem, there are no polynomial-size extended formulations achieving a $(2 - \epsilon)$-approximation [4] for any constant $\epsilon > 0$, and the standard linear programming relaxation provides a matching upper bound of 2.

Sometimes it will be convenient to work with the k-independent set polytope $\text{STAB}_k(G)$. We note that the independent set polytope $\text{STAB}(G)$ and vertex cover polytope $\text{VC}(G)$ have the same extension complexity. This can be seen by complementing the variables, i.e., replacing each instance of a variable x_i in a formulation by $1 - x_i$. The same idea shows that their cardinality-constrained counterparts $\text{VC}_k(G)$ and $\text{STAB}_{n-k}(G)$ have the same extension complexity.

Other parameters besides solution size can be used when developing extended formulations for vertex cover. A notable example is the graph invariant treewidth **tw**, which is a measure of how "tree-like" a graph is. There exist extended formulations for $\text{VC}(G)$ of size $O(2^{\mathbf{tw}} n)$ [23] (cf. an alternative proof by Buchanan and Butenko [10] based on the framework of Martin et al. [24]). Similar results hold for more general problems [5,21,22].

2. Formulations for vertex cover in hypergraphs

The main result of this section is that there are simple size $O(d^k n)$ extended formulations for k-vertex covers of d-uniform hypergraphs. This bound is $O(n)$ when $d + k$ is a constant. The extended formulations are based primarily on the following folklore lemma, which we prove for completeness.

Lemma 1 (*Folklore*). *In a d-uniform hypergraph $H = (V, E)$, the number of minimal vertex covers of H that have size $\le k$ is at most d^k.*

Proof. Denote by $S_k(H)$ the set of minimal vertex covers that have size at most k. We are to show that $|S_k(H)| \le d^k$. The proof is by induction on k. In the base case, $k = 0$. If $E = \emptyset$, then $S_0(H) = \{\emptyset\}$ so $|S_0(H)| = 1 \le d^0$. If not, then $E \ne \emptyset$ so $S_0(H) = \emptyset$ and $|S_0(H)| = 0 \le d^0$. Now suppose the claim holds for $k = p$. If $E = \emptyset$, then $|S_{p+1}(H)| = 1$. If not, let $e \in E$ and then it can be seen that $|S_{p+1}(H)| \le \sum_{j \in e} |S_p(H - j)| \le d \cdot d^p$. □

The bound in Lemma 1 is sharp. It is achieved on the d-uniform hypergraph consisting of k disjoint hyperedges.

Proposition 1. *If H is a d-uniform n-vertex hypergraph, then the k-vertex cover polytope $\text{VC}_k(H)$ of H admits an extended formulation of size $O(d^k n)$.*

Proof. Define $S_k(H)$ as in the proof of Lemma 1. For each $S \in S_k(H)$, define $P(S) := \left\{x \in [0, 1]^n \mid \sum_{i \in V} x_i = k;\ x \ge x^S\right\}$, where x^S denotes the characteristic vector of S. We claim $\text{VC}_k(H) = \text{conv.hull}\left(\cup_{S \in S_k(H)} P(S)\right) =: P$, in which case the proposition would follow by Theorem 1.

($\subseteq$) Consider an extreme point of $\text{VC}_k(H)$, which will be x^C for some k-vertex cover $C \subseteq V$. Then, C is a superset of some minimal vertex cover C', and $C' \in S_k(H)$. Thus, $x^C \in P(C') \subseteq P$.

($\supseteq$) Let x' be an extreme point of P, which will be an extreme point of some $P(S)$. It is straightforward to show that $P(S)$ is an integer polytope, so $x' = x^D$ for some $D \subseteq V$. Thus $D \supseteq S$ and $|D| = k$, meaning that D is a k-vertex cover so $x' = x^D$ belongs to $\text{VC}_k(H)$. □

We remark that a better size bound $O(f(d, k) + kn)$ can be achieved, where f is a function that depends only on d and k. To achieve this, one can exploit the "full kernel" given in Theorem 7 of Damaschke's paper [15]. We employ the full kernel in the next section, and thus do not spend time detailing the approach for hypergraphs.

3. Improved FPT formulations for vertex cover

The results of the previous section immediately imply size $O(2^k n)$ extended formulations for the k-vertex cover polytopes