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On a modification of the VCG mechanism and its optimality

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ABSTRACT

It is well-known that the VCG mechanism is optimal for a buyer procuring one unit from a set of symmetric suppliers. For procuring a unit from asymmetric suppliers, Myerson's optimal mechanism can be interpreted as a transformation of the VCG mechanism – both in terms of its allocation and payment – using the virtual cost function. For a more general setting in which multiple units need to be procured from asymmetric suppliers under an arbitrary set of feasibility constraints, we analyze the same transformation of the VCG mechanism. We show that this mechanism is optimal if the feasible region is a *polymatroid*. We also present an example of a non-polymatroidal feasible region for which this mechanism is sub-optimal.

1. Introduction

The theory of mechanism design in procurement, to a large extent, is concerned with the design of two classes of mechanisms: (1) Optimal mechanisms are the ones that maximize the expected surplus of the buyer. (2) Efficient mechanisms are those that maximize the expected social surplus, which includes both the buyer's expected surplus and the total expected surplus of the suppliers. Consider the simple setting of a buyer who wants to procure one unit of a product from a set of symmetric suppliers, each with a privately-known cost of production. It is well known that the VCG mechanism is both efficient and optimal in this setting; see e.g. [11]. In this note, we focus on the following general, constrained procurement setting:

> A buyer wants to procure multiple units of a product from either a set $\mathcal{N} = \{1, 2, \ldots, N\}$ of asymmetric suppliers, each with a privately-known unit-cost, or an outside option with unit-cost R. We denote the outside option as supplier $N + 1$. Each supplier $i \in \mathcal{N}$ has a privately-known unit-cost c_i, which is a realization of a random variable with a continuous cumulative distribution function F_i and a continuous probability density function f_i on the support $[\underline{c}_i, \bar{c}_i]$. The distributions of the costs of the suppliers are independent and common knowledge. The allocations made to the suppliers are subject to a set, denoted by FEAS, of feasibility constraints. The goal is to find an optimal mechanism for the buyer.

Feasibility constraints play an important role in procurement. This includes limitations on the allocations given to individual and/or subsets of suppliers. For example, to promote supplier diversity, AT&T [1] targets up to 15% of their total spend from minority businesses and up to 1.5% from service-disabled-veteran businesses. The restriction that a certain minimum or maximum percentage of business must be awarded to a specific class of suppliers becomes a feasibility constraint for a buying firm. Similarly, to insulate from disruption risks like those from earthquakes, many companies in the engineering sector have split their businesses across network of suppliers in multiple countries (see [12]). For strategic reasons, buying firms often impose lower and upper bounds on the number of suppliers selected and on the allocation given to each selected supplier; Mars Inc., a major producer of confectionery, uses such bounds in the procurement of its raw material (see [10]). The Hackett Group (a business advisory firm) recommends its clients to split their business among multiple vendors, giving 80% of their business to preferred vendors and the remaining 20% to backup vendors (see [13]). For other examples of several practically important feasibility constraints, we refer the readers to [2].

For every $i \in \mathcal{N}$, let $\psi_i(c) = c + F_i(c)/f_i(c)$ be a "virtual" cost function for all $c \in [\underline{c}_i, \bar{c}_i]$. We make the standard assumption that $\psi_i(\cdot)$ is strictly increasing for all $i \in \mathcal{N}$. Let $\psi_i^{-1}(v)$ be the inverse of the function $\psi_i(\cdot)$ for all $i \in \mathcal{N}$; i.e., $\psi_i^{-1}(v)$ is equal to (i) $\underline{c}_i$ if $v \le \psi_i(\underline{c}_i)$, (ii) c if $\exists c \in [\underline{c}_i, \bar{c}_i]$ such that $v = \psi_i(c)$, and (iii) $\bar{c}_i$ if $v \ge \psi_i(\bar{c}_i)$. Let $\mathbf{b} = (b_1, b_2, \ldots, b_N)$ denote the vector of bids of the suppliers, $\mathbf{b}_{-i}$ denote the vector of bids excluding the bid of supplier $i \in \mathcal{N}$, $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_{N+1})$ denote the vector of allocations given to the suppliers, and $\mathbf{M} = (M_1, M_2, \ldots, M_{N+1})$ denote the vector of payments given to the suppliers. For a bid

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vector $\mathbf{b}$, let $\boldsymbol{\psi}(\mathbf{b}) = (\psi_1(b_1), \psi_2(b_2), \dots, \psi_N(b_N))$ denote the vector of virtual bids of the suppliers.

Using standard arguments in mechanism design (see e.g., [14]), the following sealed-bid mechanism, denoted by $(\mathbf{Q}^{OPT}, \mathbf{M}^{OPT})$, is optimal under the above setting: For a given $\mathbf{b}$:

- The allocation vector $\mathbf{Q}^{OPT}(\mathbf{b}, R)$ solves:

$$\min_{\mathbf{Q}} \quad \sum_{i=1}^{N} \psi_i(b_i)Q_i + R \cdot Q_{N+1}$$
$$\text{s.t.} \quad \mathbf{Q} \in \text{FEAS}.$$

- The payment given to supplier $i \in \mathcal{N}$ is

$$M_i^{OPT}(\mathbf{b}, R) = b_i Q_i^{OPT}(\mathbf{b}, R) + \int_{b_i}^{\bar{c}_i} Q_i^{OPT}(z, \mathbf{b}_{-i}, R)\, dz \tag{M-OPT}$$

  and the payment given to supplier $N+1$ is $M_{N+1}^{OPT}(\mathbf{b}, R) = R \cdot Q_{N+1}^{OPT}(\mathbf{b}, R)$.

We refer to the above mechanism as OPT. Next, we define the VCG mechanism (see e.g., [11]), denoted by $\{\mathbf{Q}^{VCG}, \mathbf{M}^{VCG}\}$. For a given $\mathbf{b}$:

- The allocation vector $\mathbf{Q}^{VCG}(\mathbf{b}, R)$ solves:

$$\min_{\mathbf{Q}} \quad \sum_{i=1}^{N} b_i Q_i + R \cdot Q_{N+1} \tag{Q-VCG}$$
$$\text{s.t.} \quad \mathbf{Q} \in \text{FEAS}.$$

- Define $H(\mathbf{b}, R) = \sum_{i=1}^{N} b_i Q_i^{VCG}(\mathbf{b}, R) + R \cdot Q_{N+1}^{VCG}(\mathbf{b}, R)$ as the optimal value of the objective function in (Q-VCG). Let $H_{-i}(\mathbf{b}, R) = H(\mathbf{b}, R) - b_i Q_i^{VCG}(\mathbf{b}, R)$ for all $i \in \mathcal{N}$. The payment given to supplier $i \in \mathcal{N}$ is

$$M_i^{VCG}(\mathbf{b}, R) = H(\bar{c}_i, \mathbf{b}_{-i}, R) - H_{-i}(\mathbf{b}, R)$$

  and the payment given to supplier $N+1$ is $M_{N+1}^{VCG}(\mathbf{b}, R) = R \cdot Q_{N+1}^{VCG}(\mathbf{b}, R)$.

Let us now consider the special case of procuring one unit in the absence of the outside option (i.e., $R = \infty$). Let $(\tilde{1})$ and $(\tilde{2})$ denote the supplier with the lowest and second-lowest bid, respectively. Also, let (1) and (2) denote the supplier with the lowest and second-lowest virtual bid, respectively. Then, in this special case, VCG procures the unit from supplier $(\tilde{1})$ and pays him $b_{(\tilde{2})}$; i.e., the highest amount supplier $(\tilde{1})$ can bid and still be the *lowest-bid supplier*. The optimal mechanism OPT procures the unit from supplier (1) and pays him the highest amount supplier (1) can bid and still be the *lowest-virtual-bid supplier*. That is, OPT pays supplier (1) the amount $\psi_{(1)}^{-1}[\psi_{(2)}(b_{(2)})]$; this is easy to derive using (M-OPT). Motivated by this parallel between VCG and OPT in this special case, we define a transformation of VCG which we refer to as the "virtual" VCG mechanism (VVCG).

For a bid vector $\mathbf{b}$ and $i \in \mathcal{N}$, let $\boldsymbol{\psi}(\mathbf{b}) = \left(\psi_1(b_1), \psi_2(b_2), \dots, \psi_N(b_N)\right)$ denote the vector of virtual bids of the suppliers and $\hat{\boldsymbol{\psi}}_i(\mathbf{b}) = \left(\psi_i^{-1}[\psi_1(b_1)], \psi_i^{-1}[\psi_2(b_2)], \dots, \psi_i^{-1}[\psi_N(b_N)]\right)$. The VVCG mechanism, denoted by $\{\mathbf{Q}^{VVCG}, \mathbf{M}^{VVCG}\}$, is a sealed-bid mechanism defined as follows:

- The allocation given to supplier $i \in \{1, 2, \dots, N+1\}$ is

$$Q_i^{VVCG}(\mathbf{b}, R) = Q_i^{VCG}(\boldsymbol{\psi}(\mathbf{b}), R).$$

- The payment given to supplier $i \in \mathcal{N}$ is

$$M_i^{VVCG}(\mathbf{b}, R) = M_i^{VCG}(\hat{\boldsymbol{\psi}}_i(\mathbf{b}), \psi_i^{-1}(R)) \tag{M-VVCG}$$

  and the payment given to supplier $N+1$ is $M_{N+1}^{VVCG}(\mathbf{b}, R) = R \cdot Q_{N+1}^{VVCG}(\mathbf{b}, R)$.

For the special case of procuring one unit in the absence of the outside option, the VVCG mechanism simply reduces to the mechanism in which the supplier with the lowest virtual bid is selected and paid the highest amount that the supplier could bid to be selected. Thus, it is identical to OPT for this case.

VVCG-like mechanisms have precedence in the literature. For example, Hartline [9] considers a forward setting in which a seller sells multiple units of a product to a set of N bidders with a restriction that each bidder receives an allocation of either 0 or 1 unit. For this setting, the author proposes an optimal sealed-bid mechanism, which is similar to the VVCG mechanism in spirit. Another example is Duenyas et al. [4] who consider a buyer procuring an endogenously determined quantity of a product from a set of N suppliers. For this setting, the authors propose an optimal descending mechanism in which the use of virtual cost functions in designing the allocations and payments given to the suppliers is similar to that used in the VVCG mechanism.

VVCG is a natural extension of the VCG mechanism and uses virtual cost functions that play a prominent role in Myerson's seminal paper [14]. We know that VCG is an efficient mechanism. We also know from [4,9] that VVCG is optimal for the specific settings considered in those studies. Given this, it is interesting to understand if VVCG is always an optimal mechanism. This note provides a definitive answer to this question by demonstrating VVCG's optimality for a large variety of settings (i.e., polymatroid feasible allocation sets) and by showing that VVCG fails to be optimal in other settings. The discovery of this unexpected result is the main contribution of our work. Somewhat surprisingly, the proof of our result follows quite easily from existing results—thus, the proof itself is not claimed to be a significant contribution.

A closely connected paper to ours is Gupta et al. [8]. The authors of that paper propose an optimal descending mechanism for procuring multiple units of a product from a set of suppliers with a restriction of polymatroidal feasibility constraints. Their results can also be used to prove the optimality of the VVCG mechanism. In order to keep our proofs self-contained, we have presented an alternate proof for the optimality of the VVCG mechanism.

2. Optimality of VVCG

Let $g : 2^{\mathcal{N} \cup \{N+1\}} \to \mathbb{R}_+$ be a non-decreasing and submodular set function with $g(\emptyset) = 0$. That is, we have: (1) $g(\mathcal{S}) \leq g(\mathcal{T})$, for all $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{N} \cup \{N+1\}$ (non-decreasing), and (2) $g(\mathcal{S} \cup \{s\}) - g(\mathcal{S}) \geq g(\mathcal{T} \cup \{s\}) - g(\mathcal{T})$, for all $\mathcal{S} \subseteq \mathcal{T}, s \in \mathcal{N} \cup \{N+1\} \setminus \mathcal{T}$ (submodular). The set of polymatroid feasibility constraints are defined as follows:

$$P_g = \left\{ \mathbf{Q} \in R_+^{N+1} : \sum_{i \in \mathcal{S}} Q_i \leq g(\mathcal{S}) \; \forall \, \mathcal{S} \subseteq \mathcal{N} \cup \{N+1\}, \sum_{i=1}^{N+1} Q_i = g(\mathcal{N} \cup \{N+1\}) \right\}.$$

Examples of papers that consider polymatroid feasibility constraints in an auction framework include [3,7,8]. We now state our main result.

Theorem 1. *Let g be a non-decreasing and submodular set function with $g(\emptyset) = 0$. If FEAS $= P_g$, then the VVCG mechanism is identical to OPT, and therefore optimal.*

Proof. Fix an arbitrary bid vector $\mathbf{b}$. By definition, $\mathbf{Q}^{VVCG}(\mathbf{b}, R) = \mathbf{Q}^{OPT}(\mathbf{b}, R)$ and $M_{N+1}^{VVCG}(\mathbf{b}, R) = M_{N+1}^{OPT}(\mathbf{b}, R)$. Therefore, Theorem 1 can be established if we show that $M_i^{VVCG}(\mathbf{b}, R) = M_i^{OPT}(\mathbf{b}, R)$ for all $i \in \mathcal{N}$ under polymatroidal feasibility constraints. To prove this, we exploit the fact that $\mathbf{Q}^{VCG}(\boldsymbol{\psi}(\mathbf{b}), R)$ for polymatroidal feasibility constraints can be explicitly obtained by the well-known greedy