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Revenue in contests with many participants

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ABSTRACT

We show that in a contest with a single prize, the expected effort made by the kth highest valuation participant bounds the sum of the expected efforts made by all of the participants with valuations less than the kth highest valuations. We also show that in the limit case of a contest with m prizes, the expected effort made by the kth highest valuation participant when the bidders are risk-neutral is greater than the expected effort in the risk-averse case.

1. Introduction

We study a contest with m identical prizes where each participant's valuation of the prize is private information drawn independently according to the same common knowledge distribution function. In such a contest, all participants make unrecoverable efforts such as money or resources. According to the revenue equivalence theorem, if the participants are risk-neutral, the sum of the expected efforts is identical to the expected revenue in any auction mechanism such as first- or second-price auctions (see Myerson [8], Riley and Samuelson [9]). However, in first-price auctions, the winner pays the bid he or she offered, whereas in contests, the bids made by the participants are usually efforts or other non-recoverable resources. If the contest's organizer cannot recover all of the efforts made by the participants, what portion of the total efforts invested by all of the participants is recoverable?

The current study shows that when the number of participants is finite and there is a single prize, the expected payment made by the kth highest (i.e., highest with respect to effort) participant is at least twice the expected effort made by the $k+1$th participant. Thus, it follows that if the organizer can recover only k of the highest efforts, the sum of the expected efforts that cannot be recovered is bounded by the kth participant's expected effort. In other words, even if only a few of the highest efforts can be recovered, the losses are minor. This result generalizes Archak and Sundararajan [1] who show a limit result when the number of participants is infinite. In other words, when there is a single prize in the contest, in the limit case, the participant with the greatest effort generates 1/2 of the total expected efforts (Moldovanu and Sela [7] proved the result for the participants with the greatest effort), the second generates 1/4, the third 1/8, etc. Note that by the revenue equivalence theorem the sum of all of the expected efforts approaches 1 when the number of participants approaches infinity. In addition, we propose a simpler proof for the limit case than the one presented by Archak and Sundararajan [1]. Archak and Sundararajan [1] also demonstrate the existence and uniqueness of the equilibrium bid function based on mechanism design considerations. Moreover, they show that when risk aversion increases, the optimal number of prizes is more than one. We use this simpler proof to prove a generalized result for the limit case with $m \geq 1$ prizes and risk-averse participants. We show that in a contest with risk-averse participants, the expected effort made by the kth highest participant is bounded by the expected effort made by the kth participants in the risk-neutral case.

Although the results in this paper are new to the literature, it is worth noting two other studies with a different setting that is similar to the setting we present in the current study. Chawla, Hartline and Sivan [3] consider an all-pay auction mechanism where the prize is divisible. They determine that in all possible (prize divisible) all-pay auctions mechanisms, the revenue maximizing auction is the one where the participant with the greatest effort wins the entire prize. Chawla, Hartline and Sivan [3] also show that the sum

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of the expected efforts made by the participants (i.e., the seller's expected revenue) is bounded by twice the expected highest effort. In the current study we will extend this result and give a bound to the ratio between the kth highest effort and the $k+1$th highest effort. Cavallo and Jain [2] consider a different setting where the value for the contest organizer (seller) depends on the winner's personal skill, his or her level of effort and the organizer's personal quality parameters. They look for a mechanism that maximizes the social payoff, namely, the maximum expected quality for the organizer less the total efforts made by the participants. They also offer an efficient and incentive-compatible selling mechanism for their problem.

2. The model

We initially consider a contest with m identical prizes and n risk-neutral participants, each one of whom has a unit demand. Later on, we will consider risk-averse participants. Each participant has a private valuation v for a prize that has been drawn independently from a continuously differentiable distribution function $F(v)$ over the support $[0, 1]$ with a strictly positive density $F' = f > 0$. Moreover, the valuations are the private information of the participants. Participant i makes an effort b_i (e.g., resources, effort, etc.) independent of other participants. The m participants with the highest effort win a single prize, but all of the n participants pay their effort. To find the symmetric Bayesian equilibrium effort function, we follow the standard arguments (see, for example, Krishna [6]). Assume that there exists a symmetric and monotonic equilibrium effort function $b(v)$. The expected payoff for a participant with value v when he or she is playing $\hat{v}$ as his or her valuation, and all other $n-1$ participants are playing according to the equilibrium effort strategy $b(v)$, is given by

$$u(\hat{v}; v) = v_i G(\hat{v}) - b_i(\hat{v})$$

where $G(v)$ is the probability that in equilibrium, participant i will win one of the m prizes if his or her valuation of the prize is v. Given that the equilibrium effort function is monotonic with respect to v, $G(v)$ is the probability that the value v is one of the m highest valuations among the n participants and is given by

$$G(v) = \sum_{j=0}^{m-1} \binom{n-1}{j} F^{n-j-1}(v)(1-F(v))^j. \tag{1}$$

Thus, G is a distribution function of the mth highest valuation and its density is given by

$$G'(v) = \frac{(n-1)!}{(m-1)!(n-m-1)!} F^{n-m-1}(v)(1-F(v))^{m-1} f(v). \tag{2}$$

Following standard arguments, the equilibrium effort function is found by solving $\frac{\partial}{\partial \hat{v}} u(\hat{v}; v)\big|_{\hat{v}=v} = 0$, which yields

$$b(v) = vG(v) - \int_0^v G(s)ds. \tag{3}$$

Let $Y_{k,n}$ denote the distribution of the kth highest value of n participants (i.e., the kth order statistics). The distribution of $Y_{k,n}$ is given by

$$F_{Y_{k,n}}(v) = \sum_{i=0}^{k-1} \binom{n}{i} F^{n-i}(v)(1-F(v))^i. \tag{4}$$

3. A single prize and a finite number of participants

The organizer's expected revenue generated by the kth highest valuation participant is given by

$$R_k = \int_0^1 b(v) dF_{Y_{k,n}}(v). \tag{5}$$

The following proposition gives the bounds on R_k for finite n.

Proposition 1. *Let $m = 1$. Then,*

$$R_k \geq \frac{2n-1-k}{n-k} R_{k+1}.$$

Proof. Integrating by parts (5) and rearranging gives

$$\begin{aligned}
R_{k+1} &= b(1) - \int_0^1 b'(v) \sum_{i=0}^{k} \binom{n}{i} F^{n-i}(v)(1-F(v))^i dv \\
&= b(1) - \int_0^1 b'(v) \sum_{i=0}^{k-1} \binom{n}{i} F^{n-i}(v)(1-F(v))^i dv \\
&\quad - \int_0^1 b'(v) \binom{n}{k} F^{n-k}(v)(1-F(v))^k dv \\
&= R_k - \int_0^1 b'(v) \binom{n}{k} F^{n-k}(v)(1-F(v))^k dv \\
&= R_k - \int_0^1 (n-1) v f(v) F^{n-2}(v) \\
&\quad \times \binom{n}{k} F^{n-k}(v)(1-F(v))^k dv \\
&= R_k - \int_0^1 \frac{n-1}{n-k} v F^{n-1}(v) f(v) \frac{n!}{k!(n-k-1)!} \\
&\quad \times F^{n-k-1}(v)(1-F(v))^k dv \\
&= R_k - \int_0^1 \frac{n-1}{n-k} v F^{n-1}(v) dF_{Y_{k+1,n}}(v).
\end{aligned}$$

Observe that the quantity $vF^{n-1}(v)$ is the ex-ante willingness of a bidder with type v to pay. Obviously, it is higher than the bidder's bid $b(v)$. Thus, we have

$$\begin{aligned}
R_{k+1} &= R_k - \frac{n-1}{n-k} \int_0^1 v F^{n-1}(v) dF_{Y_{k+1,n}}(v) \\
&\leq R_k - \frac{n-1}{n-k} \int_0^1 b(v) dF_{Y_{k+1,n}}(v) = R_k - \frac{n-1}{n-k} R_{k+1}.
\end{aligned}$$

Rearranging completes the proof. □

In the following proposition we show that the tail of the lowest $n-k$ expected efforts is bounded by the kth expected effort.

Proposition 2. *Let $m = 1$. Then, the revenue generated by the kth highest effort is greater than the revenue generated by the sum of the successive efforts, namely $R_k \geq \sum_{i=k+1}^{n} R_i$.*

Proof. By Proposition 1, $R_k \geq \frac{2n-1-k}{n-k} R_{k+1} \geq 2R_{k+1}$. Thus,

$$\begin{aligned}
R_k &\geq 2R_{k+1} \geq R_{k+1} + 2R_{k+2} \geq R_{k+1} + R_{k+2} + 2R_{k+3} \\
&\geq \cdots \geq \sum_{i=k+1}^{n} R_i. \quad \square
\end{aligned}$$

The last result is based on the bound $R_k \geq 2R_{k+1}$, which is a weaker version of Proposition 1. However, the decline of R_k is sharper, particularly when k is increasing. In the extreme case, when $k = n-1$, $R_{n-1} \geq nR_n$. We conclude that if the organizer can recover just a few of the highest efforts, the sum of the efforts that are unrecoverable is minor.