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An optimal job, consumption/leisure, and investment policy

Gyoocheol Shim [a], Yong Hyun Shin [b,*]

[a] Department of Financial Engineering, Ajou University, Suwon 443749, Republic of Korea

[b] Department of Mathematics, Sookmyung Women's University, Seoul 140742, Republic of Korea

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ABSTRACT

In this paper we investigate an optimal job, consumption, and investment policy of an economic agent in a continuous and infinite time horizon. The agent's preference is characterized by the Cobb–Douglas utility function whose arguments are consumption and leisure. We use the martingale method to obtain the closed-form solution for the optimal job, consumption, and portfolio policy. We compare the optimal consumption and investment policy with that in the absence of job choice opportunities.

1. Introduction

We study an optimal job, consumption, and investment policy of an infinitely-lived economic agent whose preference is characterized by the Cobb–Douglas utility function of consumption and leisure. We consider two kinds of jobs one of which provides higher income but lower leisure than the other. We provide the closed-form solution for the optimal job, consumption, and investment policy by using the martingale and duality approach. We show that there is a threshold wealth level below which the optimally behaving agent chooses the job providing higher income, but above which he chooses the other job providing higher leisure. This is intuitively appealing since leisure is more important than income as the agent's wealth level gets higher. We show that the agent in our model consumes less (resp. more) when the agent's wealth is below (resp. above) the threshold level than he would if he did not have such job choice opportunities. We also show that the agent in our model takes more risk than he would without the job choice options.

There have been many extensive research studies on continuous-time portfolio selection after Merton's pioneering study (Merton [10,11]). Bodie, Merton, and Samuelson [1] have studied the effect of the labor–leisure choice on portfolio choice of an economic agent who has flexibility in his labor supply, by using the dynamic programming method. However they did not derive the closed-form solution. In this paper we use the martingale method to derive the closed-form solution. Many papers have considered portfolio selection with a retirement option: for example, Choi and Shim [2], Choi, Shim, and Shin [3], Dybvig and Liu [4], Farhi and Panageas [5], Lim and Shin [9], etc. The retirement in these papers is irreversible in that the agent cannot come back to his job after retirement, while the job choices in our model are reversible in that the agent can change the current job at any state and time.

The rest of the paper proceeds as follows. Section 2 sets up the optimization problem. Section 3 provides a solution to the problem and Section 4 investigates properties of the optimal policy.

2. The model

We consider the continuous-time financial market in an infinite-time horizon. We assume that there are two financial assets in the market: one is a riskless asset and the other is a risky asset. The risk-free interest rate $r > 0$ is assumed to be a constant and the price S_t of the risky asset is governed by the geometric Brownian motion $dS_t/S_t = \mu dt + \sigma dB_t$ for $t \geq 0$, where $(B_t)_{t=0}^{\infty}$ is a standard Brownian motion on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the parameters μ and $\sigma > 0$ are assumed to be constants. We let $\{\mathcal{F}_t\}_{t\geq 0}$ be the augmentation under $\mathbb{P}$ of the natural filtration generated by the standard Brownian motion $(B_t)_{t=0}^{\infty}$.

Let Θ_t denote the job of an economic agent at time t. The job process $\mathbf{\Theta} \triangleq (\Theta_t)_{t=0}^{\infty}$ is $\mathcal{F}_t$-adapted. For simplicity, we assume

* Corresponding author.
E-mail addresses: gshim@ajou.ac.kr (G. Shim), yhshin@sookmyung.ac.kr (Y.H. Shin).

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that there are two kinds of jobs, A_0 and A_1. The agent receives constant labor income $Y_i > 0$ and have a leisure rate L_i at each job A_i, $i = 0, 1$, where

$0 \le Y_0 < Y_1$ and $0 < L_1 < L_0$.

Let $c_t \ge 0$ and π_t denote the consumption rate and the amount of money invested in the risky asset, respectively, at time t. The consumption rate process $\mathbf{c} \triangleq (c_t)_{t=0}^{\infty}$ and the portfolio process $\boldsymbol{\pi} \triangleq (\pi_t)_{t=0}^{\infty}$ are $\mathcal{F}_t$-progressively measurable, $\int_0^t c_s ds < \infty$ for all $t \ge 0$ almost surely (a.s.), and $\int_0^t \pi_s^2 ds < \infty$ for all $t \ge 0$ a.s.

Thus the agent's wealth process $(X_t)_{t=0}^{\infty}$ with $X_0 = x$ evolves according to

$$dX_t = \left[rX_t + (\mu - r)\pi_t - c_t + Y_0 \mathbf{1}_{\{\Theta_t = A_0\}} + Y_1 \mathbf{1}_{\{\Theta_t = A_1\}}\right] dt + \sigma \pi_t dB_t. \tag{2.1}$$

The present value of the future labor income stream is Y_i/r for $\Theta_t = A_i$ where $i = 0, 1$. Since $Y_1/r > Y_0/r$ and the job state process $\boldsymbol{\Theta}$ is chosen endogenously by the agent, we let $X_0 = x > -Y_1/r$ and the agent faces the following wealth constraint:

$$X_t \ge -\frac{Y_1}{r}, \quad \text{for all } t \ge 0 \text{ a.s.} \tag{2.2}$$

We call a triple of control $(\boldsymbol{\Theta}, \mathbf{c}, \boldsymbol{\pi})$ satisfying the above conditions including (2.2) with $X_0 = x > -Y_1/r$ admissible at x. Let $\mathcal{A}(x)$ be the set of all admissible policies.

We assume that the agent has the Cobb–Douglas utility function $u(c_t, l_t)$, as in Farhi and Panageas [5]:

$$u(c_t, l_t) \triangleq \frac{1}{\alpha} \frac{\left(c_t^{\alpha} l_t^{1-\alpha}\right)^{1-\gamma}}{1-\gamma}, \quad 0 < \alpha < 1 \text{ and } 0 < \gamma \ne 1, \tag{2.3}$$

where γ is the agent's coefficient of relative risk aversion, α is a constant, and l_t is the leisure rate at time t. Let $\gamma_1 \triangleq 1 - \alpha(1-\gamma)$, and then $0 < \gamma_1 \ne 1$ and the Cobb–Douglas utility function $u(\cdot, \cdot)$ in (2.3) can be rewritten as

$$u(c_t, l_t) = l_t^{\gamma_1 - \gamma} \frac{c_t^{1-\gamma_1}}{1-\gamma_1}.$$

Remark 2.1. If $\gamma > 1$, then $\gamma > \gamma_1 > 1$, $\frac{\gamma_1}{1-\gamma_1} < 0$ and $L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} - L_1^{\frac{\gamma_1-\gamma}{\gamma_1}} < 0$. If $0 < \gamma < 1$, then $0 < \gamma < \gamma_1 < 1$, $\frac{\gamma_1}{1-\gamma_1} > 0$ and $L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} - L_1^{\frac{\gamma_1-\gamma}{\gamma_1}} > 0$. Thus the following inequality always holds:

$$\frac{\gamma_1}{1-\gamma_1}\left(L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} - L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}\right) > 0.$$

Problem 2.1. The agent's optimization problem is to maximize the expected utility

$$J(x; \boldsymbol{\Theta}, \mathbf{c}, \boldsymbol{\pi}) = \mathbb{E}\left[\int_0^{\infty} e^{-\rho t}\left(L_0^{\gamma_1-\gamma} \frac{c_t^{1-\gamma_1}}{1-\gamma_1} \mathbf{1}_{\{\Theta_t = A_0\}} + L_1^{\gamma_1-\gamma} \frac{c_t^{1-\gamma_1}}{1-\gamma_1} \mathbf{1}_{\{\Theta_t = A_1\}}\right) dt\right],$$

over $(\boldsymbol{\Theta}, \mathbf{c}, \boldsymbol{\pi}) \in \mathcal{A}(x)$, where $\rho > 0$ is a subjective discount factor.

Thus the value function $V(x)$ is given by

$$V(x) = \sup_{(\boldsymbol{\Theta}, \mathbf{c}, \boldsymbol{\pi}) \in \mathcal{A}(x)} J(x; \boldsymbol{\Theta}, \mathbf{c}, \boldsymbol{\pi}).$$

Assumption 2.1. We assume, as in Farhi and Panageas [5], that

$$K_1 \triangleq r + \frac{\rho - r}{\gamma_1} + \frac{\gamma_1 - 1}{2\gamma_1^2}\theta^2 > 0,$$

where $\theta \triangleq (\mu - r)/\sigma$, called the market price of risk.

3. The solution to the optimization problem

We denote the state price density by H_t:

$$H_t \triangleq e^{-\left(r + \frac{1}{2}\theta^2\right)t - \theta B_t}.$$

For any fixed $T \in [0, \infty)$, we denote the equivalent martingale measure by $\widetilde{\mathbb{P}}^T$:

$$\widetilde{\mathbb{P}}^T(A) = \mathbb{E}\left[e^{-\frac{1}{2}\theta^2 T - \theta B_T} \mathbf{1}_A\right], \quad \text{for } A \in \mathcal{F}_T.$$

By the Girsanov theorem, the new process $\widetilde{B}_t = B_t + \theta t$ is a standard Brownian motion for $t \in [0, T]$ under the measure $\widetilde{\mathbb{P}}^T$. As shown in Proposition 7.4 in Section 1.7 of Karatzas and Shreve [7], there exists a unique probability measure $\widetilde{\mathbb{P}}$ on $\mathcal{F}_\infty$ which agrees with $\widetilde{\mathbb{P}}^T$ on $\mathcal{F}_T$, for $T \in [0, \infty)$, and $\widetilde{B}_t$ is a standard Brownian motion for $t \in [0, \infty)$ under $\widetilde{\mathbb{P}}$. Thus Eq. (2.1) can be rewritten as

$$dX_t = \left[rX_t - c_t + Y_0 \mathbf{1}_{\{\Theta_t = A_0\}} + Y_1 \mathbf{1}_{\{\Theta_t = A_1\}}\right] dt + \sigma \pi_t d\widetilde{B}_t. \tag{3.1}$$

By (2.2) and (3.1), we derive, similarly to Lim and Shin [9], the following budget constraint:

$$\mathbb{E}\left[\int_0^{\infty} \left(c_t - Y_0 \mathbf{1}_{\{\Theta_t = A_0\}} - Y_1 \mathbf{1}_{\{\Theta_t = A_1\}}\right) H_t dt\right] \le x.$$

For a Lagrange multiplier $\lambda > 0$, a dual value function is

$$\begin{aligned}
\widetilde{V}(\lambda) + \lambda x
&= \sup_{(\boldsymbol{\Theta}, \mathbf{c}, \boldsymbol{\pi}) \in \mathcal{A}(x)} \mathbb{E}\left[\int_0^{\infty} e^{-\rho t}\left(L_0^{\gamma_1-\gamma} \frac{c_t^{1-\gamma_1}}{1-\gamma_1} \mathbf{1}_{\{\Theta_t = A_0\}} + L_1^{\gamma_1-\gamma} \frac{c_t^{1-\gamma_1}}{1-\gamma_1} \mathbf{1}_{\{\Theta_t = A_1\}}\right) dt \right.\\
&\quad \left. - \lambda \int_0^{\infty} \left(c_t - Y_0 \mathbf{1}_{\{\Theta_t = A_0\}} - Y_1 \mathbf{1}_{\{\Theta_t = A_1\}}\right) H_t dt\right] + \lambda x \\
&= \mathbb{E}\left[\int_0^{\infty} e^{-\rho t}\left(\widetilde{u}_0(z_t) \mathbf{1}_{\{0 < z_t \le \bar{z}\}} + \widetilde{u}_1(z_t) \mathbf{1}_{\{z_t > \bar{z}\}}\right) dt\right] + \lambda x,
\end{aligned} \tag{3.2}$$

if the job and consumption strategy $(\Theta_t^{\lambda}, c_t^{\lambda})$ is given by

$$\Theta_t^{\lambda} = \begin{cases} A_0, & \text{if } 0 < z_t \le \bar{z}, \\ A_1, & \text{if } z_t > \bar{z}, \end{cases}$$

$$c_t^{\lambda} = \begin{cases} L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} (z_t)^{-\frac{1}{\gamma_1}}, & \text{if } 0 < z_t \le \bar{z}, \\ L_1^{\frac{\gamma_1-\gamma}{\gamma_1}} (z_t)^{-\frac{1}{\gamma_1}}, & \text{if } z_t > \bar{z}, \end{cases}$$

where

$$\begin{aligned}
\widetilde{u}_i(z) &= \sup_{c \ge 0}\left(L_i^{\gamma_1-\gamma} \frac{c^{1-\gamma_1}}{1-\gamma_1} - cz\right) + Y_i z \\
&= L_i^{\frac{\gamma_1-\gamma}{\gamma_1}} \frac{\gamma_1}{1-\gamma_1} z^{-\frac{1-\gamma_1}{\gamma_1}} + Y_i z, \quad i = 0, 1,
\end{aligned}$$

$$z_t \triangleq \lambda e^{\rho t} H_t = \lambda e^{\left(\rho - r - \frac{1}{2}\theta^2\right)t - \theta B_t}, \tag{3.3}$$

and $\bar{z}$ is the solution to the algebraic equation $\widetilde{u}_0(z) = \widetilde{u}_1(z)$:

$$\bar{z} = \left(\frac{\frac{\gamma_1}{1-\gamma_1}\left(L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} - L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}\right)}{Y_1 - Y_0}\right)^{\gamma_1} > 0, \tag{3.4}$$

which is positive by Remark 2.1. Similarly to Proposition 6.5 in Karatzas and Wang [8], the Lagrange multiplier λ is chosen in (3.12) so that

$$\mathbb{E}\left[\int_0^{\infty} \left(c_t^{\lambda} - Y_0 \mathbf{1}_{\{0 < z_t \le \bar{z}\}} - Y_1 \mathbf{1}_{\{z_t > \bar{z}\}}\right) H_t dt\right] = x \tag{3.5}$$