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On a cardinality-constrained transportation problem with market choice

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ABSTRACT

It is well-known that the intersection of the matching polytope with a cardinality constraint is integral (Schrijver, 2003) [8]. In this note, we prove a similar result for the polytope corresponding to the transportation problem with market choice (TPMC) (introduced in Damcı-Kurt et al. (2015)) when the demands are in the set $\{1, 2\}$. This result generalizes the result regarding the matching polytope. The result in this note implies that some special classes of minimum weight perfect matching problem with a cardinality constraint on a subset of edges can be solved in polynomial time.

1. Introduction

The transportation problem with market choice (TPMC), introduced in the paper [4], is a transportation problem in which suppliers with limited capacities have a choice of which demands (markets) to satisfy. If a market is selected, then its demand must be satisfied fully through shipments from the suppliers. If a market is rejected, then the corresponding potential revenue is lost. The objective is to minimize the total cost of shipping and lost revenues. See [5,7,9] for approximation algorithms and heuristics for several other supply chain planning and logistics problems with market choice.

Formally, we are given a set of supply and demand nodes that form a bipartite graph $G = (V_1 \cup V_2, E)$. The nodes in set V_1 represent the supply nodes, where for $i \in V_1$, $s_i \in \mathbb{N}$ represents the capacity of supplier i. The nodes in set V_2 represent the potential markets, where for $j \in V_2$, $d_j \in \mathbb{N}$ represents the demand of market j. The edges between supply and demand nodes have weights that represent shipping costs w_e, where $e \in E$. For each $j \in V_2$, r_j is the revenue lost if the market j is rejected. Let $x_{\{i,j\}}$ be the amount of demand of market j satisfied by supplier i for $\{i, j\} \in E$, and let z_j be an indicator variable taking a value 1 if market j is rejected and 0 otherwise. A mixed-integer programming (MIP) formulation of the problem is given where the objective is to minimize the transportation costs and the lost revenues due to unchosen markets:

$$\min_{x\in\mathbb{R}^{|E|}_+,\, z\in\{0,1\}^{|V_2|}} \quad \sum_{e\in E} w_e x_e + \sum_{j\in V_2} r_j z_j \tag{1}$$

$$\text{s.t.} \quad \sum_{i:\{i,j\}\in E} x_{\{i,j\}} = d_j(1-z_j) \qquad \forall j \in V_2 \tag{2}$$

$$\sum_{j:\{i,j\}\in E} x_{\{i,j\}} \le s_i \qquad \forall i \in V_1. \tag{3}$$

We refer to the formulation (1)–(3) as TPMC. The first set of constraints (2) ensures that if market $j \in V_2$ is selected (i.e., $z_j = 0$), then its demand must be fully satisfied. The second set of constraints (3) model the supply restrictions.

TPMC is strongly NP-complete in general [4]. Aardal and Le Bodic [1] give polynomial-time reductions from this problem to the capacitated facility location problem [6], thereby establishing approximation algorithms with constant factors for the metric case and a logarithmic factor for the general case.

When $d_j \in \{1, 2\}$ for each demand node $j \in V_2$, TPMC is polynomially solvable [4]. We call this special class of the problem, the *simple TPMC problem* in the rest of this note.

Observation 1 (*Simple TPMC Generalizes Matching on General Graphs*)**.** *The matching problem can be seen as a special case of the simple TPMC problem. Let $G = (V, E)$ be a graph with n vertices and*

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m edges. We construct a bipartite graph $\hat{G} = (\hat{V}^1 \cup \hat{V}^2, \hat{E})$ as follows: $\hat{V}^1$ is a set of n vertices corresponding to the n vertices in G, and $\hat{V}^2$ corresponds to the set of edges of G, i.e., $\hat{V}^2$ contains m vertices. We use $\{i, j\}$ to refer to the vertex in $\hat{V}^2$ corresponding to the edge $\{i, j\}$ in E. The set of edges in $\hat{E}$ are of the form $\{i, \{i, j\}\}$ and $\{j, \{i, j\}\}$ for every $i, j \in V$ such that $\{i, j\} \in E$. Now we can construct (the feasible region of) an instance of TPMC with respect to $\hat{G} = (\hat{V}^1 \cup \hat{V}^2, \hat{E})$ as follows:

$$T = \{(x, z) \in \mathbb{R}^{2m}_+ \times \mathbb{R}^m \mid x_{\{i,e\}} + x_{\{j,e\}} + 2z_e = 2 \ \forall e = \{i, j\} \in \hat{V}^2 \quad (4)$$

$$\sum_{j:\{i,j\}\in E} x_{\{i,\{i,j\}\}} \leq 1 \ \forall i \in \hat{V}^1 \quad (5)$$

$$z_e \in \{0, 1\} \ \forall e \in \hat{V}^2\}. \quad (6)$$

Clearly there is a bijection between the set of matchings in G and the set of solutions in T. Moreover, let

$$H := \{(x, z, y) \in \mathbb{R}^{2m} \times \mathbb{R}^m \times \mathbb{R}^m \mid (x, z) \in T, y = e - z\},$$

where e is the all ones vector in $\mathbb{R}^m$. Then we have that the convex hull of the incidence vectors of all the matchings in $G = (V, E)$ is precisely the set $\mathrm{proj}_y(H)$.

Note that the instances of the form of (4)–(6) are special cases of simple TPMC instances, since in these instances all s_i's are restricted to be exactly 1 and all d_j's are restricted to be exactly 2.

2. Main result

An important and natural constraint that one may add to the TPMC problem is that of a service level, that is the number of rejected markets is restricted to be at most k. This restriction can be modeled using a *cardinality constraint*, $\sum_{j\in V_2} z_j \leq k$, appended to (1)–(3). We call the resulting problem cardinality-constrained TPMC (CCTPMC). If we are able to solve CCTPMC in polynomial-time, then we can solve TPMC in polynomial time by solving CCTPMC for all $k \in \{0, \dots, |V_2|\}$. Since TPMC is NP-hard, CCTPMC is NP-hard in general.

In this note, we examine the effect of appending a cardinality constraint to the simple TPMC problem.

Theorem 1. *Given an instance of TPMC with V_2, the set of demand nodes, and E, the set of edges, let $X \subseteq \mathbb{R}^{|E|}_+ \times \{0, 1\}^{|V_2|}$ be the set of feasible solutions of the simple TPMC. Let $k \in \mathbb{Z}_+$ and $k \leq |V_2|$. Let $X^k := \mathrm{conv}(X \cap \{(x, z) \in \mathbb{R}^{|E|}_+ \times \{0, 1\}^{|V_2|} \mid \sum_{j\in V_2} z_j \leq k\})$. If $d_j \leq 2$ for all $j \in V_2$, then $X^k = \mathrm{conv}(X) \cap \{(x, z) \in \mathbb{R}^{|E|}_+ \times [0, 1]^{|V_2|} \mid \sum_{j\in V_2} z_j \leq k\}$.*

Our proof of Theorem 1 is presented in Section 3. We note that the result of Theorem 1 holds even when X^k is defined as $\mathrm{conv}(X \cap \{(x, z) \in \mathbb{R}^{|E|}_+ \times \{0, 1\}^{|V_2|} \mid \sum_{j\in V_2} z_j \geq k\})$ or $\mathrm{conv}(X \cap \{(x, z) \in \mathbb{R}^{|E|}_+ \times \{0, 1\}^{|V_2|} \mid \sum_{j\in V_2} z_j = k\})$.

In Lemma 1 in Section 3, we give a linear description of $\mathrm{conv}(X)$ by means of a projection of a matching polytope over which we can optimize in polynomial time. Therefore, by invoking the ellipsoid algorithm and the use of Theorem 1 we obtain the following corollary.

Corollary 1. *Cardinality constrained simple TPMC is polynomially solvable.*

We note that, as a consequence of Theorem 1 (but also inherent in our proof), a special class of minimum weight perfect matching problem with a cardinality constraint on a subset of edges can be solved in polynomial time: Simple TPMC can be reduced to a minimum weight perfect matching problem on a general (non-bipartite) graph $G' = (V', E')$ [4]. (Note that Observation 1, in contrast, provides a reduction *from* matching *to* a special case of simple TPMC.) Therefore, it is possible to reduce CCTPMC with $d_j \leq 2$ for all $j \in V_2$ to a *minimum weight perfect matching problem with a cardinality constraint on a subset of edges*. Hence, Corollary 1 implies that a special class of minimum weight perfect matching problems with a cardinality constraint on a subset of edges can be solved in polynomial time.

Note that the intersection of the perfect matching polytope with a cardinality constraint on a strict subset of edges is not always integral.

Example 1. Consider the cycle C_4 of length 4 with edge set $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$, and the cardinality constraint $x_{12} + x_{34} = 1$. The only perfect matchings are $\{\{1, 2\}, \{3, 4\}\}$ and $\{\{1, 4\}, \{2, 3\}\}$ for which the cardinality constraint has activity 2 and 0, respectively. Thus the perfect matching polytope is a line which is intersected by the hyperplane defined by the cardinality constraint in the (fractional) center.

To the best of our knowledge, the complexity status of minimum weight perfect matching problem on a general graph with a cardinality constraint on a subset of edges is open. This can be seen by observing that if one can solve minimum weight perfect matching problem with a cardinality constraint on a subset of edges in polynomial time, then one can solve the exact perfect matching problem, in polynomial time. Given a weighted graph, *the exact perfect matching problem* is to find a perfect matching that has a total weight equal to a given number. The complexity status of exact perfect matching is open; see discussion in the last section in [2].

Finally we ask the natural question: Does the statement of Theorem 1 hold when $d_j \leq 2$ does not hold for every j? The next example illustrates that the statement does not hold in such case.

Example 2. Consider an instance of TPMC where $G = (V_1 \cup V_2, E)$ is a bipartite graph with

$$V_1 = \{i_1, i_2, i_3, i_4, i_5, i_6\}, \qquad V_2 = \{j_1, j_2, j_3, j_4\},$$
$$E = \{\{i_1, j_1\}, \{i_2, j_2\}, \{i_3, j_3\}, \{i_4, j_1\}, \{i_4, j_4\}, \{i_5, j_2\}, \{i_5, j_4\}, \{i_6, j_3\}, \{i_6, j_4\}\},$$
$$s_i = 1, \quad i \in V_1, \qquad d_{j_1} = d_{j_2} = d_{j_3} = 2, \qquad d_{j_4} = 3.$$

For $k = 2$ it can be verified that we obtain a non-integer extreme point of $\mathrm{conv}(X) \cap \{(x, z) \in \mathbb{R}^p_+ \times [0, 1]^n \mid \sum_{j=1}^n z_j \leq k\}$, given by $x_{\{i_1,j_1\}} = x_{\{i_2,j_2\}} = x_{\{i_3,j_3\}} = x_{\{i_4,j_1\}} = x_{\{i_4,j_4\}} = x_{\{i_5,j_2\}} = x_{\{i_5,j_4\}} = x_{\{i_6,j_3\}} = x_{\{i_6,j_4\}} = z_1 = z_2 = z_3 = z_4 = \frac{1}{2}$. To see this, consider the face defined by the supply constraints of nodes $\{i_4, i_5, i_6\}$ and observe that this face has precisely two solutions having 1 and 3 markets, respectively.

Therefore, $X^k \neq \mathrm{conv}(X) \cap \{(x, z) \in \mathbb{R}^p_+ \times [0, 1]^n \mid \sum_{j=1}^n z_j \leq k\}$ in this example.

3. Proof of Theorem 1

To prove Theorem 1 we use an improved reduction to a minimum weight matching problem (compared to the reduction in [4]) and then use the well-known adjacency properties of the vertices of the perfect matching polytope. Since the integrality result does not hold for the perfect matching polytope on a general graph with a cardinality constraint on any subset of edges, as illustrated in Example 1, we need to refine the adjacency criterion.

We begin with some notation. For a graph $G = (V, E)$ with node set V and edge set E, and a node $v \in V$, we denote by $\delta(v) := \delta_G(v) := \{e \in E \mid v \in e\}$ the set of edges incident to